

STABILITY OF THE TRAVELING WAVES FOR THE DERIVATIVE SCHRÖDINGER EQUATION IN THE ENERGY SPACE

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ABSTRACT. In this paper, we continue the study of the dynamics of the traveling waves for nonlinear Schrödinger equation with derivative (DNLS) in the energy space. Under some technical assumptions on the speed of each traveling wave, the stability of the sum of two traveling waves for DNLS is obtained in the energy space by Martel-Merle-Tsai's analytic approach in [20, 21]. As a by-product, we also give an alternative proof of the stability of the single traveling wave in the energy space in [6], where Colin and Ohta made use of the concentration-compactness argument.

1. INTRODUCTION

In this paper, we consider the Cauchy problem for nonlinear Schrödinger equation with derivative (DNLS)

$$\begin{cases} i\partial_t v + \partial_x^2 v + i\partial_x (|v|^2 v) = 0, & t \in \mathbb{R}, \\ v(0, x) = v_0(x) \in H^1(\mathbb{R}). \end{cases} \quad (1.1)$$

(1.1) appears in plasma physics [24, 25, 32]. There are many equivalent forms under the gauge transformation. For instance, if we take the following gauge transformation,

$$v(t, x) \mapsto u(t, x) = G_{3/4}(v)(t, x) \triangleq e^{i\frac{3}{4} \int_{-\infty}^x |v(t, \eta)|^2 d\eta} v(t, x),$$

then (1.1) is equivalent to the following equation (DNLS)

$$\begin{cases} i\partial_t u + \partial_x^2 u + \frac{1}{2}i|u|^2 \partial_x u - \frac{1}{2}iu^2 \partial_x \bar{u} + \frac{3}{16}|u|^4 u = 0, & t \in \mathbb{R} \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}), \end{cases} \quad (1.2)$$

which is L^2 -critical NLS with derivative for the fact that the scaling symmetry

$$u(t, x) \mapsto u_\lambda(t, x) = \lambda^{1/2} u(\lambda^2 t, \lambda x)$$

leaves both (1.2) and the mass invariant. The mass, momentum and energy are defined as following

$$\begin{aligned} M(u)(t) &= \frac{1}{2} \int |u(t, x)|^2 dx, \\ P(u)(t) &= -\frac{1}{2} \Im \int (\bar{u} \partial_x u)(t, x) dx + \frac{1}{8} \int |u(t, x)|^4 dx, \end{aligned}$$

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$$E(u)(t) = \frac{1}{2} \int |\partial_x u(t, x)|^2 dx - \frac{1}{32} \int |u(t, x)|^6 dx.$$

They are conserved under the flow (1.2) according to the phase rotation invariance, spatial translation invariance and time translation invariance respectively. Compared with the L^2 -critical NLS, (1.1) or (1.2) doesn't enjoy the Galilean invariance and pseudo-conformal invariance any more.

Local well-posedness result for (1.2) in the energy space has been worked out by Hayashi and Ozawa [14, 27]. They combined the fixed point argument with $L_I^4 W^{1,\infty}(\mathbb{R})$ estimate to construct the local-in-time solution with arbitrary data in the energy space. For other kinds of local well-posedness results, we can refer to [12, 13]. Since (1.2) is \dot{H}^1 -subcritical, the maximal lifespan interval only depends on the H^1 norm of initial data. More precisely, we have

Theorem 1.1. [14, 27] *For any $u_0 \in H^1(\mathbb{R})$ and $t_0 \in \mathbb{R}$, there exists a unique maximal-lifespan solution $u : I \times \mathbb{R} \rightarrow \mathbb{C}$ to (1.2) with $u(t_0) = u_0$, the map $u_0 \rightarrow u$ is continuous from $H^1(\mathbb{R})$ to $C(I, H^1(\mathbb{R})) \cap L_{loc}^4(I; W^{1,\infty}(\mathbb{R}))$. Moreover, the solution also has the following properties:*

- (1) I is an open neighborhood of t_0 .
- (2) The mass, momentum and energy are conserved, that is, for all $t \in I$,

$$M(u)(t) = M(u)(t_0), \quad P(u)(t) = P(u)(t_0), \quad E(u)(t) = E(u)(t_0).$$

- (3) If $\sup(I) < +\infty$, (or $\inf(I) > -\infty$), then

$$\lim_{t \rightarrow \sup(I)} \|\partial_x u(t)\|_{L^2} = +\infty, \quad \left(\lim_{t \rightarrow \inf(I)} \|\partial_x u(t)\|_{L^2} = +\infty, \text{ respectively.} \right)$$

- (4) If $\|u_0\|_{H^1}$ is sufficiently small, then u is a global solution.

The sharp local well-posedness result in H^s , $s \geq 1/2$ is due to Takaoka [33] by using Bourgain's space. The sharpness is shown in [34] in the sense that nonlinear evolution $u(0) \mapsto u(t)$ fails to be C^3 or even uniformly C^0 in this topology, even when t is arbitrarily close to zero and H^s norm of the data is small (see also Biagioni-Linares [3]).

After the sharp local wellposedness is obtained, there are two aspects of global solution of (1.2) to be concerned. One is about the global wellposedness in the lower regularity space $H^s(\mathbb{R})$ for some $s < 1$, another one is about the dynamics of the traveling waves in the energy space $H^1(\mathbb{R})$.

On one hand, the global well-posedness is obtained for (1.2) in the energy space in [27] under the smallness condition

$$\|u_0\|_{L^2} < \sqrt{2\pi}, \tag{1.3}$$

the argument is based on the energy method (conservation of mass and energy) together with the sharp Gagliardo-Nirenberg inequality [36]. This is improved by Takaoka [34], who proved global well-posedness in H^s for $s > 32/33$ under the condition (1.3). His argument is based on

Bourgain's restriction method, which separated the evolution of low frequencies and of high frequencies of initial data and noticed that the nonlinear evolution has H^1 regularity effect even for the rough solution $u \in H^s$, $s < 1$. In [7, 8], I-team used the "I-method" to show global well-posedness in H^s , $s > 1/2$ under (1.3), I-team defined Iu as a modified function, whose energy is nearly conserved in time by capturing nonlinear cancellation in frequency space under the flow (1.2). Later, Miao, Wu and Xu [23] showed the sharp global well-posedness in $H^{1/2}$ under (1.3) by using I-method together with the refined resonant decomposition. For the global result, we can also refer to the recent paper [18, 29] by the inverse scattering method.

On the other hand, it is known in [15, 22, 31] that (1.2) has a two-parameter family of the traveling waves with the form:

$$u(t, x) \triangleq e^{i\omega t} \varphi_{\omega, c}(x - ct) \quad (1.4)$$

$$\triangleq e^{i\omega t + i\frac{c}{2}(x-ct)} \phi_{\omega, c}(x - ct) \quad (1.5)$$

where $(\omega, c) \in \mathbb{R}^2$, $c^2 < 4\omega$ and

$$\phi_{\omega, c}(x) = \left[\frac{\sqrt{\omega}}{4\omega - c^2} \left\{ \cosh(\sqrt{4\omega - c^2}x) - \frac{c}{2\sqrt{\omega}} \right\} \right]^{-1/2}, \quad (1.6)$$

which is a positive solution of

$$\left(\omega - \frac{c^2}{4} \right) \phi - \partial_x^2 \phi - \frac{3}{16} |\phi|^4 \phi = -\frac{c}{2} |\phi|^2 \phi. \quad (1.7)$$

In fact, by (1.2) and (1.4), we know that the solitary solution $\varphi_{\omega, c}$ should satisfy the following equation

$$\omega \varphi - \partial_x^2 \varphi - \frac{3}{16} |\varphi|^4 \varphi = -ic \partial_x \varphi + \frac{1}{2} i |\varphi|^2 \partial_x \varphi - \frac{1}{2} i \varphi^2 \partial_x \bar{\varphi}. \quad (1.8)$$

In [22], the authors characterized all solutions to (1.8) in the energy space, which corresponds to all traveling waves to (1.2).

Theorem 1.2. *Let*

$$\phi_{\omega, c}(x) = \begin{cases} \left[\frac{\sqrt{\omega}}{4\omega - c^2} \left\{ \cosh(\sqrt{4\omega - c^2}x) - \frac{c}{2\sqrt{\omega}} \right\} \right]^{-1/2}, & c^2 < 4\omega \\ 2\sqrt{c} \cdot (c^2 x^2 + 1)^{-1/2}, & c^2 = 4\omega, c > 0. \end{cases}$$

Then the following results hold

- (1) *For the subcritical case $c^2 < 4\omega$. $\varphi_{\omega, c}(x) = e^{i\frac{c}{2}x} \phi_{\omega, c}(x)$ is the unique solution of (1.8) among nontrivial solution in $H^1(\mathbb{R})$, up to the phase rotation and spatial translation symmetries of (1.8).*
- (2) *For the critical case $c^2 = 4\omega$, $c > 0$. $\varphi_{\omega, c}(x) = e^{i\frac{c}{2}x} \phi_{\omega, c}(x)$ is the unique solution of (1.8) among nontrivial solution in $H^1(\mathbb{R})$, up to the phase rotation and spatial translation symmetries of (1.8).*

(3) For the critical case $c^2 = 4\omega, c \leq 0$ and the supercritical case $c^2 > 4\omega$. (1.8) has no nontrivial solution in $H^1(\mathbb{R})$.

Remark 1.3. The key observation to the above result is to make use of the structure of solution to (1.8) according to (1.5), it is equivalent to solve a semilinear ODE with the Nehari manifold argument and the non-increasing rearrangement technique [1, 2, 17, 39]. As a consequence of the variational characterization of $\varphi_{\omega,c}(x)$, a sufficient condition on the global wellposedness of the solution $u(t, x)$ to (1.2) was shown in H^1 in [22]. That is, if the initial data $u_0 \in H^1(\mathbb{R})$ satisfies¹ $J_{\omega,c}(u_0) < J_{\omega,c}(\varphi_{\omega,c})$, $K_{\omega,c}(u_0) \geq 0$ for some (ω, c) with $c^2 < 4\omega$ or $c^2 = 4\omega, c > 0$, then the solution $u(t)$ to (1.2) exists globally in time. While there is no blowup result for the solution $u(t)$ with $J_{\omega,c}(u_0) < J_{\omega,c}(\varphi_{\omega,c})$, $K_{\omega,c}(u_0) < 0$ in $H^1(\mathbb{R})$ because of the lack of the effective Virial identity.

In this paper, we will focus on the study of the stability of the traveling waves in the energy space. For the subcritical case $c^2 < 4\omega$, Colin and Ohta made use of the concentration compactness argument to show the stability of the single traveling wave for (1.2) in [6], which extended the result in [11]. It is noticed that Martel, Merle and Tsai developed some powerful analytic approach to show the stability of the multi-soliton wave for subcritical gKdV and NLS in [20, 21], which is based on the modulation analysis [37, 38], perturbation theory, monotonicity formulas and the conservation laws. For the orbital stability results, we can also refer to [4, 5, 9, 10, 26, 28, 35]. Here we will apply this analytic method to (1.2) and obtain the following results. First of all, we revisit the stability of the single traveling wave for (1.2) in the energy space.

Theorem 1.4. For any $(\omega^0, c^0) \in \mathbb{R}^2$ with $(c^0)^2 < 4\omega^0$, the traveling wave solution $e^{i\omega^0 t} \varphi_{\omega^0, c^0}(x - c^0 t)$ to (1.2) is orbitally stable in the energy space. That is, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R})$ satisfies

$$\|u_0(\cdot) - \varphi_{\omega^0, c^0}(\cdot - x^0)e^{i\gamma^0}\|_{H^1(\mathbb{R})} < \delta$$

for some $(x^0, \gamma^0) \in \mathbb{R}^2$, then the solution $u(t)$ of (1.2) exists globally in time and satisfies

$$\sup_{t \geq 0} \inf_{(y, \gamma) \in \mathbb{R}^2} \|u(t, \cdot) - \varphi_{\omega^0, c^0}(\cdot - y)e^{i\gamma}\|_{H^1(\mathbb{R})} < \epsilon.$$

Remark 1.5. (1) About the dynamics of the solution to (1.2) around the traveling waves, the above result extends the results in [22, 40].

(2) As for the critical parameters $c^2 = 4\omega, c > 0$, we don't know whether the corresponding traveling waves for (1.2) in [22] is stable or not since there is in lack of the spectral gap for the spectrum of the linearized operator around the traveling waves.

¹Here the minimum action functional $J_{\omega,c}$ and the scaling derivative functional $K_{\omega,c}$ are defined in $H^1(\mathbb{R})$ as following

$$\begin{aligned} J_{\omega,c}(\varphi) &\triangleq E(\varphi) + \omega M(\varphi) + cP(\varphi), \\ K_{\omega,c}(\varphi) &\triangleq \int \left(|\varphi_x|^2 - \frac{3}{16} |\varphi|^6 + \omega |\varphi|^2 - c \Im(\overline{\varphi} \varphi_x) + \frac{c}{2} |\varphi|^4 \right) dx. \end{aligned}$$

Secondly, we can show the stability of the sum of two traveling waves for (1.2) in the energy space when the centers of two traveling waves are always far away (that is, weak interaction) from each other. That is,

Theorem 1.6. *Let $(\omega_k^0, c_k^0) \in \mathbb{R}^2$, $k = 1, 2$ satisfy*

- (a) *Nonlinear stability of each wave: $(c_k^0)^2 < 4\omega_k^0$ for $k = 1, 2$.*
- (b) *Forward propagation of each wave: $0 < c_1^0 < c_2^0$.*
- (c) *Relative speed: $\max(c_1^0, 0) < 2 \frac{\omega_2^0 - \omega_1^0}{c_2^0 - c_1^0} < c_2^0$.*

Then there exists $C, \delta_0, \theta_0, L_0 > 0$, such that if $0 < \delta < \delta_0$, $L > L_0$ and

$$\left\| u_0(\cdot) - \sum_{k=1}^2 \varphi_{\omega_k^0, c_k^0}(\cdot - x_k^0) e^{i\gamma_k^0} \right\|_{H^1(\mathbb{R})} \leq \delta,$$

with $x_2^0 - x_1^0 > L$, then the solution $u(t)$ of (1.2) exists globally in time and there exist \mathcal{C}^1 functions $x_k(t)$ and $\gamma_k(t)$, $k = 1, 2$ such that for any $t \geq 0$,

$$\left\| u(t, \cdot) - \sum_{k=1}^2 \varphi_{\omega_k^0, c_k^0}(\cdot - x_k(t)) e^{i\gamma_k(t)} \right\|_{H^1(\mathbb{R})} \leq C \left(\delta + e^{-\theta_0 \frac{L}{2}} \right).$$

Remark 1.7. (1) Since the traveling wave with $(c_k^0)^2 < 4\omega_k^0$ is orbital stable by Theorem 1.4, we call the condition $(c_k^0)^2 < 4\omega_k^0$ *nonlinear stable condition*.

- (2) About assumption (b): Forward propagation of each wave. It is a special case for the two-traveling wave without collisions, the arguments can be extended to other cases without collisions: (I) : $c_1^0 < 0 < c_2^0$ (The right one propagates forward, the left one propagate backward). From the view of physics, it is more stable since two traveling waves propagate in different directions. (II) : $c_1^0 < c_2^0 < 0$ (Backward propagation of each wave). It is the same as the case we consider by symmetry. In fact, it is more easier than the case we consider since we needn't modulate the speed parameter c_k in the modulation analysis.
- (3) About assumption (c): Relative speed. Compared with the single traveling wave case, the multi-traveling wave case is more technical. In fact, after the modulation analysis of the solution around the sum of two traveling waves, we use the technical assumption (c) for some monotonicity formulas to show the localized action functional $\mathfrak{E}(u(t))$ is almost conserved, and obtain the refined estimates of the parameters $|\omega_k(t) - \omega_k(0)| + |c_k(t) - c_k(0)|$, $k = 1, 2$. Please refer Section 6 to the details.
- (4) About the the k -soliton case ($k \geq 3$), Le Coz and Wu [16] obtained the stability of a k -soliton solution of (DNLS) independently a few months after we submitted our paper. The main difference is that Le Coz and Wu considered the $\frac{1}{2}$ -gauge transformed version of (1.1), whereas our paper is based on the form (1.2), which corresponds to the $\frac{3}{4}$ -gauge transformed version of (1.1).

At last, the paper is organized as following. In Section 2, we introduce the linearized operator around single traveling wave, show the coercivity property of the linearized energy and obtain the geometric decomposition of the solution around single traveling wave. In Section 3, inspired by the ideas in [21], on one hand, we introduce a conserved functional, which is related to the mass, momentum and energy, to obtain a refined estimate about the remainder term in the modulation analysis (geometric decomposition) of the solution. On the other hand, we use the conservation laws of mass and momentum to refine the estimate of the parameters $|\omega(t) - \omega(0)| + |c(t) - c(0)|$. Together with the continuity argument, these refined estimates imply Theorem 1.4. In Section 4, we give the modulation analysis of the solution around the sum of two traveling waves with weak interactions. In Section 5, we introduce some extra monotonicity formulas and their dynamics. In Section 6, on one hand, we introduce a localized action functional, which is almost conserved by the monotonicity formula and the conservation laws of mass, momentum and energy, to refine the estimate about the remainder term in the modulation analysis of the solution. On the other hand, we use some monotonicity formulas to refine the estimates of $|\omega_k(t) - \omega_k(0)| + |c_k(t) - c_k(0)|$, $k = 1, 2$ besides of the conservation laws of mass and momentum. These refined estimates also imply Theorem 1.6 together with the continuity argument.

In Appendix A, we give the coercivity of the quadratic term (i.e. the linearized energy) under the orthogonal structures. In Appendix B, we use the perturbation theory to linearize the action functional of the solution around the sum of two traveling waves. In Appendix C, we show the coercivity properties of the localized quadratic term under the orthogonal structures.

2. PROPERTIES OF THE LINEARIZED OPERATOR AROUND THE TRAVELING WAVE

In this section, we will describe some basic properties of the traveling wave with the sub-critical parameters $c^2 < 4\omega$ for (1.2).

When $c^2 < 4\omega$, it is well known that (1.2) has the following kinds of traveling wave solutions (for instance, see [15, 22, 31])

$$u_{\omega,c}(t, x) \triangleq e^{i\omega t} \varphi_{\omega,c}(x - ct) \quad (2.1)$$

$$\triangleq e^{i\omega t + i\frac{c}{2}(x-ct)} \phi_{\omega,c}(x - ct) \quad (2.2)$$

where

$$\phi_{\omega,c}(x) = \left[\frac{\sqrt{\omega}}{4\omega - c^2} \left\{ \cosh(\sqrt{4\omega - c^2}x) - \frac{c}{2\sqrt{\omega}} \right\} \right]^{-1/2}, \quad (2.3)$$

which is the unique positive solution up to the symmetries of (1.7).² By the concentration-compactness argument in [6], we know that the traveling waves (2.1) with $c^2 < 4\omega$ are orbitally stable. It is worth noticing that the condition $c^2 < 4\omega$ implies the following non-degenerate

²For $(\omega, c) \in \mathbb{R}^2$, we have characterized all solutions for (1.8) in Theorem 1.2.

condition [6]

$$\det d''(\omega, c) < 0, \quad (2.4)$$

where $d(\omega, c) \triangleq J_{\omega, c}(\varphi_{\omega, c}) = E(\varphi_{\omega, c}) + \omega M(\varphi_{\omega, c}) + cP(\varphi_{\omega, c})$. In fact, since $\varphi_{\omega, c}$ is the critical point of the action functional $J_{\omega, c}(\varphi)$ in [22], we have

$$\partial_{\omega} d(\omega, c) = M(\varphi_{\omega, c}), \quad \partial_c d(\omega, c) = P(\varphi_{\omega, c}),$$

and

$$d''(\omega, c) = \begin{pmatrix} \partial_{\omega} M(\varphi_{\omega, c}) & \partial_c M(\varphi_{\omega, c}) \\ \partial_{\omega} P(\varphi_{\omega, c}) & \partial_c P(\varphi_{\omega, c}) \end{pmatrix}.$$

2.1. Linearized operator and its coercivity. We now consider the linearized operator around the traveling waves and its coercivity in $H^1(\mathbb{R})$ in this subsection.

Let $c^2 < 4\omega$, we define the operators for any real valued functions $\eta_1, \eta_2 \in H^1(\mathbb{R})$

$$\begin{aligned} \mathcal{L}_+ \eta_1 &\triangleq -\frac{1}{2} \partial_x^2 \eta_1 + \frac{1}{2} \left(\omega - \frac{c^2}{4} \right) \eta_1 + \frac{3}{4} c \phi_{\omega, c}^2 \eta_1 - \frac{15}{32} \phi_{\omega, c}^4 \eta_1, \\ \mathcal{L}_- \eta_2 &\triangleq -\frac{1}{2} \partial_x^2 \eta_2 + \frac{1}{2} \left(\omega - \frac{c^2}{4} \right) \eta_2 + \frac{1}{4} c \phi_{\omega, c}^2 \eta_2 - \frac{3}{32} \phi_{\omega, c}^4 \eta_2, \end{aligned}$$

and the quadratic form for any complex valued functions $\eta = \eta_1 + i\eta_2 \in H^1(\mathbb{R})$

$$\begin{aligned} \tilde{\mathcal{H}}_{\omega, c}(\eta, \eta) &\triangleq (\mathcal{L}_+ \eta_1, \eta_1) + (\mathcal{L}_- \eta_2, \eta_2) \\ &= \int \frac{1}{2} |\partial_x \eta|^2 + \frac{1}{2} \left(\omega - \frac{1}{4} c^2 \right) |\eta|^2 \, dx \\ &\quad + \int \left(-\frac{3}{32} \phi_{\omega, c}^4 |\eta|^2 - \frac{3}{8} \phi_{\omega, c}^4 |\eta_1|^2 + \frac{1}{4} c \phi_{\omega, c}^2 |\eta|^2 + \frac{1}{2} c \phi_{\omega, c}^2 |\eta_1|^2 \right) \, dx. \end{aligned}$$

The next result shows that for the subcritical case, the coercivity of the linearized operator around the traveling wave holds under some orthogonal conditions in the energy space.

Proposition 2.1. *Assume that $c^2 < 4\omega$, then the following results hold*

(1) *If $\psi \in H^1(\mathbb{R})$ satisfies $(\psi, \phi_{\omega, c}) = 0$, then there exists $C_1 > 0$ such that*

$$(\mathcal{L}_- \psi, \psi) \geq C_1 \|\psi\|_2^2.$$

(2) *If $\psi \in H^1(\mathbb{R})$ satisfies $(\psi, \phi_{\omega, c}) = 0$, $(\psi, \phi_{\omega, c}^3) = 0$, and $(\psi, \partial_x \phi_{\omega, c}) = 0$, then there exists $C_2 > 0$ such that*

$$(\mathcal{L}_+ \psi, \psi) \geq C_2 \|\psi\|_2^2.$$

The proof of Proposition 2.1 follows from a standard variational argument. We refer to Appendix A for the proof. As a consequence, we have

Proposition 2.2. Assume that $c^2 < 4\omega$. Let $\eta \in H^1(\mathbb{R})$ be such that $(\Im \eta, \phi_{\omega,c}) = 0$, and

$$(\Re \eta, \phi_{\omega,c}) = 0, \quad (\Re \eta, \partial_x \phi_{\omega,c}) = 0, \quad \frac{1}{2} (\Re \eta, \phi_{\omega,c}^3) + (\Im \eta, \partial_x \phi_{\omega,c}) = 0,$$

then there exists $C > 0$ such that

$$\tilde{\mathcal{H}}_{\omega,c}(\eta, \eta) \geq C \|\eta\|_{H^1(\mathbb{R})}^2. \quad (2.5)$$

Proof. Let $\eta_1 = \Re \eta$ and $\eta_2 = \Im \eta$. In order to prove (2.5), it suffices to show that

$$\tilde{\mathcal{H}}_{\omega,c}(\eta, \eta) \geq C \|\eta\|_2^2. \quad (2.6)$$

On one hand, by Proposition 2.1 and

$$(\eta_1, \phi_{\omega,c}) = 0, \quad (\eta_1, \partial_x \phi_{\omega,c}) = 0,$$

there exist constants $c, \tilde{C}_1 > 0$ such that

$$\|\eta_1\|_2^2 \leq \frac{1}{c} (\mathcal{L}_+ \eta_1, \eta_1) + \tilde{C}_1 (\eta_1, \phi_{\omega,c}^3)^2. \quad (2.7)$$

Now inserting

$$\frac{1}{2} (\eta_1, \phi_{\omega,c}^3) + (\eta_2, \partial_x \phi_{\omega,c}) = 0$$

into (2.7) and by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|\eta_1\|_2^2 &\leq \frac{1}{c} (\mathcal{L}_+ \eta_1, \eta_1) + 4\tilde{C}_1 (\eta_2, \partial_x \phi_{\omega,c})^2 \\ &\leq \frac{1}{c} (\mathcal{L}_+ \eta_1, \eta_1) + \tilde{C}_2 \|\eta_2\|_2^2. \end{aligned} \quad (2.8)$$

On the other hand, by Proposition 2.1 and $(\eta_2, \phi_{\omega,c}) = 0$, there exists a constant $C_1 > 0$ such that

$$C_1 \|\eta_2\|_2^2 \leq (\mathcal{L}_- \eta_2, \eta_2). \quad (2.9)$$

Without loss of generality, we can choose $0 < c \ll 1$ with $c(1 + \tilde{C}_2) \leq C_1$ by Proposition 2.1. By (2.8) and (2.9), we have

$$\begin{aligned} \|\eta_1\|_2^2 + \|\eta_2\|_2^2 &\leq \frac{1}{c} (\mathcal{L}_+ \eta_1, \eta_1) + \tilde{C}_2 \|\eta_2\|_2^2 + \frac{1}{C_1} (\mathcal{L}_- \eta_2, \eta_2) \\ &\leq \frac{1}{c} (\mathcal{L}_+ \eta_1, \eta_1) + \left(\frac{\tilde{C}_2}{C_1} + \frac{1}{C_1} \right) (\mathcal{L}_- \eta_2, \eta_2) \\ &\leq \frac{1}{c} ((\mathcal{L}_+ \eta_1, \eta_1) + (\mathcal{L}_- \eta_2, \eta_2)). \end{aligned}$$

This can complete the proof of (2.6) by the definition of $\tilde{\mathcal{H}}_{\omega,c}$. \square

2.2. Decomposition of the functions close to a traveling wave. Let ω^0, c^0 satisfy $(c^0)^2 < 4\omega^0$. For $\alpha > 0$, we consider the following tube of size α in H^1 ,

$$\mathcal{U}_1(\alpha, \omega^0, c^0) \triangleq \left\{ u \in H^1(\mathbb{R}) : \inf_{(x, \gamma) \in \mathbb{R}^2} \|u(\cdot) - \varphi_{\omega^0, c^0}(\cdot - x) e^{i\gamma}\|_{H^1(\mathbb{R})} < \alpha \right\},$$

which is close to some traveling wave with the subcritical parameter. First, by the non-degenerate condition on $d''(\omega^0, c^0)$, we have the following structure decomposition for the functions in the above tube by the implicit function theorem.

Lemma 2.3. *There exists $\alpha_0 > 0, C_I > 0$, such that if $u \in \mathcal{U}_1(\delta, \omega^0, c^0)$ with $\delta < \alpha_0$, then there exist unique C^1 functions*

$$\vec{q}_0 \triangleq (\omega_0, c_0, x_0, \gamma_0) \in (0, +\infty) \times \mathbb{R}^3$$

with $c_0^2 < 4\omega_0$, such that

$$\Re \int R(x) \overline{\varepsilon(x)} dx = 0, \quad \Re \int \left(i\partial_x R + \frac{1}{2} |R|^2 R \right) (x) \overline{\varepsilon(x)} dx = 0, \quad (2.10)$$

$$\Re \int \partial_x R(x) \overline{\varepsilon(x)} dx = 0, \quad \Re \int iR(x) \overline{\varepsilon(x)} dx = 0. \quad (2.11)$$

where

$$R(x) = R(\omega_0, c_0, x_0, \gamma_0; x) \triangleq \varphi_{\omega_0, c_0}(x - x_0) e^{i\gamma_0}, \quad (2.12)$$

$$\varepsilon(x) = \varepsilon(\omega_0, c_0, x_0, \gamma_0, u; x) \triangleq u(x) - R(\omega_0, c_0, x_0, \gamma_0; x). \quad (2.13)$$

Moreover, we have

$$\|\varepsilon\|_{H^1(\mathbb{R})} + |\omega_0 - \omega^0| + |c_0 - c^0| \leq C_I \delta.$$

Remark 2.4. The orthogonal structures in (2.10) correspond to the conservation laws of mass and momentum, while the orthogonal structures in (2.11) correspond to the spatial translation invariance and the phase rotation invariance.

Proof of Lemma 2.3. For any $u \in \mathcal{U}_1(\delta, \omega^0, c^0)$, there exists $(x^0, \gamma^0) \in \mathbb{R}^2$ such that

$$\left\| u(\cdot) - \varphi_{\omega^0, c^0}(\cdot - x^0) e^{i\gamma^0} \right\|_{H^1(\mathbb{R})} < \delta.$$

Now let

$$\vec{q}^0 = (\omega^0, c^0, x^0, \gamma^0), \text{ and } R^0(x) = \varphi_{\omega^0, c^0}(x - x^0) e^{i\gamma^0},$$

and define

$$\varrho_1(\omega_0, c_0, x_0, \gamma_0, u) \triangleq \Re \int R(\omega_0, c_0, x_0, \gamma_0; x) \overline{\varepsilon(\omega_0, c_0, x_0, \gamma_0, u; x)} dx,$$

$$\varrho_2(\omega_0, c_0, x_0, \gamma_0, u) \triangleq \Re \int \left(i\partial_x R + \frac{1}{2} |R|^2 R \right) (\omega_0, c_0, x_0, \gamma_0; x) \overline{\varepsilon(\omega_0, c_0, x_0, \gamma_0, u; x)} dx,$$

$$\varrho_3(\omega_0, c_0, x_0, \gamma_0, u) \triangleq \Re \int \partial_x R(\omega_0, c_0, x_0, \gamma_0; x) \overline{\varepsilon(\omega_0, c_0, x_0, \gamma_0, u; x)} dx,$$

$$\varrho_4(\omega_0, c_0, x_0, \gamma_0, u) \triangleq \Re \int iR(\omega_0, c_0, x_0, \gamma_0; x) \overline{\varepsilon(\omega_0, c_0, x_0, \gamma_0, u; x)} dx,$$

where $R(\omega_0, c_0, x_0, \gamma_0; x)$ and $\varepsilon(\omega_0, c_0, x_0, \gamma_0, u; x)$ are defined by (2.12) and (2.13). By the direct calculations, we have

$$\varepsilon(\vec{q}^0, R^0; x) = 0;$$

and

$$\begin{aligned} \left(\frac{\partial}{\partial \omega_0} \varepsilon \right) (\vec{q}^0, R^0; x) &= - \frac{\partial}{\partial \omega_0} R \Big|_{\vec{q}^0 = \vec{q}^0}, & \left(\frac{\partial}{\partial c_0} \varepsilon \right) (\vec{q}^0, R^0; x) &= - \frac{\partial}{\partial c_0} R \Big|_{\vec{q}^0 = \vec{q}^0}, \\ \left(\frac{\partial}{\partial x_0} \varepsilon \right) (\vec{q}^0, R^0; x) &= - \frac{\partial}{\partial x_0} R \Big|_{\vec{q}^0 = \vec{q}^0}, & \left(\frac{\partial}{\partial \gamma_0} \varepsilon \right) (\vec{q}^0, R^0; x) &= -iR \Big|_{\vec{q}^0 = \vec{q}^0}. \end{aligned}$$

These imply that at the point (\vec{q}^0, R^0) ,

$$\begin{aligned} & \frac{\partial(\varrho_1, \varrho_2, \varrho_3, \varrho_4)}{\partial(\omega_0, c_0, x_0, \gamma_0)} \Big|_{(\vec{q}^0, u)=(\vec{q}^0, R^0)} \\ &= \begin{pmatrix} -\partial_{\omega_0} M(R) & -\partial_{c_0} M(R) & 0 & 0 \\ -\partial_{\omega_0} P(R) & -\partial_{c_0} P(R) & 0 & 0 \\ -\Re \int \partial_x R \overline{\partial_{\omega_0} R} & -\Re \int \partial_x R \overline{\partial_{c_0} R} & -\Re \int \partial_x R \overline{\partial_{x_0} R} & -\Re \int \partial_x R \overline{iR} \\ -\Re \int iR \overline{\partial_{\omega_0} R} & -\Re \int iR \overline{\partial_{c_0} R} & -\Re \int iR \overline{\partial_{x_0} R} & -\Re \int iR \overline{iR} \end{pmatrix} \Big|_{(\vec{q}^0, u)=(\vec{q}^0, R^0)} \end{aligned}$$

By the simple calculations, the determinant of the above Jacobian is

$$-\|\partial_x \phi_{\omega^0, c^0}\|_2^2 \cdot \|\phi_{\omega^0, c^0}\|_2^2 \cdot \det d''(\omega^0, c^0),$$

which is non-degenerate for $(c^0)^2 < 4\omega^0$. Thus we can complete the proof by the implicit function theorem. \square

3. STABILITY OF THE SINGLE TRAVELING WAVE

In this section, we shall give an alternative proof of the orbital stability of the single traveling wave with $c^2 < 4\omega$ in the energy space in [6], where the argument is based on the concentration compactness principle. Now inspired by Martel-Merle-Tsai's idea in [21], our argument is the energy method together with the modulation analysis and perturbation theory, which can be applied to the multi-traveling wave case with the weak interactions.

Let $(\omega^0, c^0) \in \mathbb{R}^2$ satisfy $(c^0)^2 < 4\omega^0$, α_0 be determined by Lemma 2.3, $A_0, \delta_0 = \delta_0(A_0)$ be determined later, and $\delta < \alpha_0$. Suppose that $u(t)$ is the solution of (1.2) with the initial data $u_0 \in \mathcal{U}_1(\delta, \omega^0, c^0)$, then by the definition of the small tube $\mathcal{U}_1(\delta, \omega^0, c^0)$, there exist $x^0 \in \mathbb{R}$ and $\gamma^0 \in \mathbb{R}$ such that

$$\left\| u_0(\cdot) - \varphi_{\omega^0, c^0}(\cdot - x^0) e^{i\gamma^0} \right\|_{H^1(\mathbb{R})} < \delta.$$

Let $A_0 > 2$ be determined later and define

$$T^* = \sup \{ t \geq 0, \sup_{\tau \in [0, t]} \inf_{x^0 \in \mathbb{R}, \gamma^0 \in \mathbb{R}} \left\| u(\tau, \cdot) - \varphi_{\omega^0, c^0}(\cdot - x^0) e^{i\gamma^0} \right\|_{H^1(\mathbb{R})} \leq A_0 \delta \}.$$

By the continuity of $u(t)$ in H^1 , we know that $T^* > 0$. In order to prove Theorem 1.4, it suffices to show $T^* = +\infty$ for some constants $\delta_0 > 0$ and $A_0 > 2$.

Assume that $T^* < +\infty$, we know that for any $t \in [0, T^*]$, there exist $(x^0(t), \gamma^0(t)) \in \mathbb{R}^2$ such that

$$\left\| u(t, \cdot) - \varphi_{\omega^0, c^0}(\cdot - x^0(t)) e^{i\gamma^0(t)} \right\|_{H^1(\mathbb{R})} \leq A_0 \delta.$$

If necessary, we can choose δ_0 sufficiently small to ensure that the condition $A_0 \delta_0 < \alpha_0$ holds, which enables us to establish the structure decomposition to the solution $u(t)$, $t \in [0, T^*]$.

Step 1: The geometric decomposition of solution $u(t)$ around the single traveling wave. By Lemma 2.3, we can modify the parameters $\omega^0, c^0, x^0(t)$ and $\gamma^0(t)$ such that $(c(t))^2 < 4\omega(t)$ for any $t \in [0, T^*]$, and the remainder term

$$\varepsilon(t, x) \triangleq u(t, x) - R(t, x), \quad \text{where } R(t, x) \triangleq \varphi_{\omega(t), c(t)}(x - x(t)) e^{i\gamma(t)}, \quad (3.1)$$

has the following orthogonal structures

$$\Re \int R(t) \overline{\varepsilon(t)} dx = 0, \quad \Re \int \left(i \partial_x R(t) + \frac{1}{2} |R(t)|^2 R(t) \right) \overline{\varepsilon(t)} dx = 0, \quad (3.2)$$

$$\Re \int \partial_x R(t) \overline{\varepsilon(t)} dx = 0, \quad \Re \int i R(t) \overline{\varepsilon(t)} dx = 0, \quad (3.3)$$

and for any $t \in [0, T^*]$, we have

$$\|\varepsilon(0)\|_{H^1(\mathbb{R})} + |\omega(0) - \omega^0| + |c(0) - c^0| \leq C_I \delta, \quad (3.4)$$

$$\|\varepsilon(t)\|_{H^1(\mathbb{R})} + |\omega(t) - \omega^0| + |c(t) - c^0| \leq C_I A_0 \delta. \quad (3.5)$$

If necessary, we can choose δ_0 sufficiently small such that $C_I A_0 \delta_0 < 1$. Since (3.5) is too rough, we will combine the energy method with the coercivity property of the linearized operator to show more refined estimates.

Step 2: A conserved functional and refined estimate of the remainder term $\|\varepsilon(t)\|_{H^1(\mathbb{R})}$. We now introduce the following functional with parameters $\omega(0)$ and $c(0)$

$$\mathfrak{J}_{\omega(0), c(0)}(u(t)) \triangleq E(u(t)) + \omega(0) \cdot M(u(t)) + c(0) \cdot P(u(t)), \quad (3.6)$$

which is conserved for the solution $u(t)$ of (1.2) by the conservation laws of mass, momentum and energy.

By the decomposition (3.1), the orthogonal structures (3.2) and the estimate (3.5), we have the following expansion formula.

Lemma 3.1.

$$\begin{aligned} \mathfrak{J}_{\omega(0),c(0)}(u(t)) &= \mathfrak{J}_{\omega(0),c(0)}(R(0)) + \mathcal{H}_{\omega(t),c(t)}(\varepsilon(t), \varepsilon(t)) + \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \beta\left(\|\varepsilon(t)\|_{H^1(\mathbb{R})}\right) \\ &\quad + O\left(|\omega(t) - \omega(0)|^2 + |c(t) - c(0)|^2\right), \end{aligned} \quad (3.7)$$

where $\beta(r) \rightarrow 0$ as $r \rightarrow 0$, and

$$\begin{aligned} \mathcal{H}_{\omega(t),c(t)}(\varepsilon(t), \varepsilon(t)) &\triangleq \int \frac{1}{2} |\partial_x \varepsilon(t)|^2 - \frac{3}{32} |R(t)|^4 |\varepsilon(t)|^2 - \frac{3}{8} |R(t)|^2 \left(\Re(\overline{R(t)} \varepsilon(t)) \right)^2 dx \\ &\quad + \int \frac{1}{2} \omega(t) |\varepsilon(t)|^2 - \frac{1}{2} c(t) \Im(\overline{\varepsilon(t)} \partial_x \varepsilon(t)) dx \\ &\quad + \int \frac{1}{4} c(t) |R(t)|^2 |\varepsilon(t)|^2 dx + \frac{1}{2} c(t) \left(\Re(\overline{R(t)} \varepsilon(t)) \right)^2 dx. \end{aligned}$$

Proof. Firstly, by (3.1), (3.5) and the integration by parts, we have

$$\begin{aligned} \frac{1}{2} \int |\partial_x u(t)|^2 dx &= \frac{1}{2} \int |\partial_x R(t)|^2 - 2 \Re(\overline{\partial_x^2 R(t)} \varepsilon(t)) + |\partial_x \varepsilon(t)|^2 dx, \\ -\frac{1}{32} \int |u(t)|^6 dx &= -\frac{1}{32} \int |R(t)|^6 + 6 |R(t)|^4 \Re(\overline{R(t)} \varepsilon(t)) + 3 |R(t)|^4 |\varepsilon(t)|^2 dx \\ &\quad - \frac{1}{32} \int 12 |R(t)|^2 \left(\Re(\overline{R(t)} \varepsilon(t)) \right)^2 dx + \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \beta\left(\|\varepsilon(t)\|_{H^1(\mathbb{R})}\right). \end{aligned}$$

Secondly, by the formula of mass $M(u(t))$, we have

$$\frac{1}{2} \int |u(t)|^2 dx = \frac{1}{2} \int |R(t)|^2 + 2 \Re(\overline{R(t)} \varepsilon(t)) + |\varepsilon(t)|^2 dx.$$

Last, by the formula of momentum $P(u(t))$, we get

$$\begin{aligned} -\frac{1}{2} \Im \int \overline{u(t)} \partial_x u(t) dx &= -\frac{1}{2} \Im \int \overline{R(t)} \partial_x R(t) + \overline{R(t)} \partial_x \varepsilon(t) + \overline{\varepsilon(t)} \partial_x R(t) + \overline{\varepsilon(t)} \partial_x \varepsilon(t) dx \\ &= -\frac{1}{2} \Im \int \overline{R(t)} \partial_x R(t) + \overline{\varepsilon(t)} \partial_x \varepsilon(t) dx - \Re \int i \overline{\partial_x R(t)} \varepsilon(t) dx, \\ \frac{1}{8} \int |u(t)|^4 dx &= \frac{1}{8} \int |R(t)|^4 + 4 |R(t)|^2 \Re(\overline{R(t)} \varepsilon(t)) + 2 |R(t)|^2 |\varepsilon(t)|^2 dx \\ &\quad + \frac{1}{8} \int 4 \left(\Re(\overline{R(t)} \varepsilon(t)) \right)^2 dx + \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \beta\left(\|\varepsilon(t)\|_{H^1(\mathbb{R})}\right). \end{aligned}$$

Multiplying $M(u(t))$ with $\omega(0)$ and $P(u(t))$ with $c(0)$ and summing up, we obtain

$$\begin{aligned} &\mathfrak{J}_{\omega(0),c(0)}(u(t)) \\ &= \mathfrak{J}_{\omega(0),c(0)}(R(t)) \\ &\quad + \Re \int \left(-\overline{\partial_x^2 R} - \frac{3}{16} |R|^4 \overline{R} + \omega(0) \overline{R} - ic(0) \overline{\partial_x R} + \frac{1}{2} c(0) |R|^2 \overline{R} \right) (t) \varepsilon(t) dx \\ &\quad + \int \frac{1}{2} |\partial_x \varepsilon(t)|^2 - \frac{3}{32} |R(t)|^4 |\varepsilon(t)|^2 R(t) dx - \frac{3}{8} |R(t)|^2 \left(\Re(\overline{R(t)} \varepsilon(t)) \right)^2 dx \\ &\quad + \int \frac{1}{2} \omega(0) |\varepsilon(t)|^2 - \frac{1}{2} c(0) \Im(\overline{\varepsilon(t)} \partial_x \varepsilon(t)) dx \end{aligned}$$

$$\begin{aligned}
& + \int \frac{1}{4} c(0) |R(t)|^2 |\varepsilon(t)|^2 dx + \frac{1}{2} c(0) \left(\Re \overline{R(t)} \varepsilon(t) \right)^2 dx \\
& + \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \beta \left(\|\varepsilon(t)\|_{H^1(\mathbb{R})} \right).
\end{aligned} \tag{3.8}$$

We first deal with the linear term in $\varepsilon(t)$. Since $R(t, x) = \varphi_{\omega(t), c(t)}(x - y(t)) e^{i\gamma(t)}$ satisfies that the following elliptic equation

$$-\overline{\partial_x^2 R(t)} - \frac{3}{16} |R(t)|^4 \overline{R(t)} + \omega(t) \overline{R(t)} - ic(t) \overline{\partial_x R(t)} + \frac{1}{2} c(t) |R(t)|^2 \overline{R(t)} = 0, \tag{3.9}$$

we have the following cancelation by the orthogonal structure (3.2)

$$\begin{aligned}
& \Re \int \left(-\overline{\partial_x^2 R(t)} - \frac{3}{16} |R(t)|^4 \overline{R(t)} + \omega(t) \overline{R(t)} - ic(t) \overline{\partial_x R(t)} + \frac{1}{2} c(t) |R(t)|^2 \overline{R(t)} \right) \varepsilon(t) dx \\
& = (\omega(0) - \omega(t)) \Re \int \overline{R(t)} \varepsilon(t) dx + (c(0) - c(t)) \Re \int \left(-i \overline{\partial_x R(t)} + \frac{1}{2} |R(t)|^2 \overline{R(t)} \right) \varepsilon(t) dx \\
& = (\omega(0) - \omega(t)) \Re \int R(t) \overline{\varepsilon(t)} dx + (c(0) - c(t)) \Re \int \left(i \partial_x R(t) + \frac{1}{2} |R(t)|^2 R(t) \right) \overline{\varepsilon(t)} dx \\
& = 0.
\end{aligned} \tag{3.10}$$

Secondly, by the definition of the linearized energy $\mathcal{H}_{\omega(t), c(t)}(\varepsilon(t), \varepsilon(t))$, we have

$$\begin{aligned}
& \int \frac{1}{2} |\partial_x \varepsilon(t)|^2 - \frac{3}{32} |R(t)|^4 |\varepsilon(t)|^2 R(t) dx - \frac{3}{8} |R(t)|^2 \left(\Re \left(\overline{R(t)} \varepsilon(t) \right) \right)^2 dx \\
& + \int \frac{1}{2} \omega(0) |\varepsilon(t)|^2 - \frac{1}{2} c(0) \Im \left(\overline{\varepsilon(t)} \partial_x \varepsilon(t) \right) dx \\
& + \int \frac{1}{4} c(0) |R(t)|^2 |\varepsilon(t)|^2 dx + \frac{1}{2} c(0) \left(\Re \left(\overline{R(t)} \varepsilon(t) \right) \right)^2 dx \\
& = \mathcal{H}_{\omega(t), c(t)}(\varepsilon(t), \varepsilon(t)) + C \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 (|\omega(t) - \omega(0)| + |c(t) - c(0)|) \\
& = \mathcal{H}_{\omega(t), c(t)}(\varepsilon(t), \varepsilon(t)) + O \left(|\omega(t) - \omega(0)|^2 + |c(t) - c(0)|^2 \right) \\
& + \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \beta \left(\|\varepsilon(t)\|_{H^1(\mathbb{R})} \right).
\end{aligned} \tag{3.11}$$

Finally, we estimate the main term by Taylor's expansion.

$$\begin{aligned}
& \mathfrak{J}_{\omega(0), c(0)}(R(t)) \\
& = \mathfrak{J}_{\omega(t), c(t)}(R(t)) + (\omega(0) - \omega(t)) M(R(t)) + (c(0) - c(t)) P(R(t)) \\
& = J_{\omega(t), c(t)}(\varphi_{\omega(t), c(t)}) + (\omega(0) - \omega(t)) M(\varphi_{\omega(t), c(t)}(t)) + (c(0) - c(t)) P(\varphi_{\omega(t), c(t)}(t)) \\
& = J_{\omega(0), c(0)}(\varphi_{\omega(0), c(0)}) + (\omega(t) - \omega(0)) M(\varphi_{\omega(0), c(0)}) + (c(t) - c(0)) P(\varphi_{\omega(0), c(0)}) \\
& \quad + O \left(|\omega(t) - \omega(0)|^2 + |c(t) - c(0)|^2 \right) + (\omega(0) - \omega(t)) M(\varphi_{\omega(t), c(t)}) + (c(0) - c(t)) P(\varphi_{\omega(t), c(t)}) \\
& = J_{\omega(0), c(0)}(\varphi_{\omega(0), c(0)}) + O \left(|\omega(t) - \omega(0)|^2 + |c(t) - c(0)|^2 \right) \\
& = \mathfrak{J}_{\omega(0), c(0)}(R(0)) + O \left(|\omega(t) - \omega(0)|^2 + |c(t) - c(0)|^2 \right)
\end{aligned} \tag{3.12}$$

where we used the fact that

$$\frac{\partial}{\partial \omega} J_{\omega, c}(\varphi_{\omega, c}) = M(\varphi_{\omega, c}), \quad \frac{\partial}{\partial c} J_{\omega, c}(\varphi_{\omega, c}) = P(\varphi_{\omega, c})$$

in the third equality. Inserting (3.10), (3.11) and (3.12) into (3.8), we can complete the proof. \square

By the conservation laws of mass, momentum and energy, we have

$$\mathfrak{J}_{\omega(0), c(0)}(u(t)) = \mathfrak{J}_{\omega(0), c(0)}(u(0)).$$

This together with Lemma 3.1 and (3.4) implies that

$$\begin{aligned} & \mathcal{H}_{\omega(t), c(t)}(\varepsilon(t), \varepsilon(t)) \\ &= \mathcal{H}_{\omega(0), c(0)}(\varepsilon(0), \varepsilon(0)) + \|\varepsilon(0)\|_{H^1(\mathbb{R})}^2 \beta \left(\|\varepsilon(0)\|_{H^1(\mathbb{R})} \right) + \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \beta \left(\|\varepsilon(t)\|_{H^1(\mathbb{R})} \right) \\ & \quad + O(|\omega(t) - \omega(0)|^2 + |c(t) - c(0)|^2) \\ & \leq C \|\varepsilon(0)\|_{H^1(\mathbb{R})}^2 + \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \beta \left(\|\varepsilon(t)\|_{H^1(\mathbb{R})} \right) + O(|\omega(t) - \omega(0)|^2 + |c(t) - c(0)|^2). \end{aligned} \tag{3.13}$$

As for the linearized energy $\mathcal{H}_{\omega(t), c(t)}(\varepsilon(t), \varepsilon(t))$ with $c(t)^2 < 4\omega(t)$, we have the following coercivity property under the orthogonal conditions (3.2) and (3.3).

Lemma 3.2. *Suppose that $c(t)^2 < 4\omega(t)$ for any $t \in [0, T^*]$, and $\varepsilon(t) \in H^1(\mathbb{R})$ satisfies the orthogonal conditions (3.2) and (3.3), then there exists a constant $C_0 > 0$ such that*

$$C_0 \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \leq \mathcal{H}_{\omega(t), c(t)}(\varepsilon(t), \varepsilon(t)).$$

Proof. Since the traveling wave has the following structure

$$R(t, x) = \varphi_{\omega(t), c(t)}(x - x(t)) e^{i\gamma(t)} = \phi_{\omega(t), c(t)}(x - x(t)) e^{i\gamma(t)} e^{i\frac{1}{2}c(t)(x - x(t))},$$

we introduce the similar structure for the remainder term $\varepsilon(t)$ and define $\eta(t)$ as following

$$\eta(t, x) = \varepsilon(t, x + x(t)) e^{-i\gamma(t)} e^{-i\frac{1}{2}c(t)x} \iff \varepsilon(t, x) = \eta(t, x - x(t)) e^{i\gamma(t)} e^{i\frac{1}{2}c(t)(x - x(t))}.$$

By the simple computations, the linearized energy $\mathcal{H}_{\omega(t), c(t)}(\varepsilon(t), \varepsilon(t))$ is

$$\begin{aligned} & \mathcal{H}_{\omega(t), c(t)}(\varepsilon(t), \varepsilon(t)) \\ &= \int \left(\frac{1}{2} |\partial_x \eta|^2 + \frac{1}{2} \left(\omega(t) - \frac{1}{4} c(t)^2 \right) |\eta|^2 - \frac{3}{32} \phi_{\omega(t), c(t)}^4 |\eta|^2 - \frac{3}{8} \phi_{\omega(t), c(t)}^4 |\eta_1(t)|^2 \right) dx \\ & \quad + \int \left(\frac{1}{4} c(t) \phi_{\omega(t), c(t)}^2 |\eta(t)|^2 + \frac{1}{2} c(t) \phi_{\omega(t), c(t)}^2 |\eta_1(t)|^2 \right) dx \\ &= (\mathcal{L}_+ \eta_1(t), \eta_1(t)) + (\mathcal{L}_- \eta_2(t), \eta_2(t)) = \tilde{\mathcal{H}}_{\omega(t), c(t)}(\eta(t), \eta(t)), \end{aligned}$$

the orthogonal conditions (3.2) and (3.3) on $\varepsilon(t)$ are equivalent to the following conditions on $\eta(t) = \eta_1(t) + i\eta_2(t)$:

$$(\eta_2(t), \phi_{\omega(t), c(t)}) = 0,$$

and $(\eta_1(t), \phi_{\omega(t), c(t)}) = 0$, $(\eta_1(t), \partial_x \phi_{\omega(t), c(t)}) = 0$,

$$\frac{1}{2} \left(\eta_1(t), \phi_{\omega(t), c(t)}^3 \right) + (\eta_2(t), \partial_x \phi_{\omega(t), c(t)}) = 0.$$

By Proposition 2.2, we have

$$\mathcal{H}_{\omega(t), c(t)}(\varepsilon(t), \varepsilon(t)) = \tilde{\mathcal{H}}_{\omega(t), c(t)}(\eta(t), \eta(t)) \geq C \|\eta(t)\|_{H^1(\mathbb{R})}^2,$$

which together (3.5) and the fact that

$$\|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \leq C \|\eta_x(t)\|_2^2 + C(1 + c(t)^2) \|\eta(t)\|_2^2$$

implies the result. \square

By Lemma 3.2 and (3.13), there exists some constant $C > 0$ such that for any $t \in [0, T^*]$, we have

$$\|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \leq C \|\varepsilon(0)\|_{H^1(\mathbb{R})}^2 + C \left(|\omega(t) - \omega(0)|^2 + |c(t) - c(0)|^2 \right). \quad (3.14)$$

This completes the refined estimate of the remainder term $\|\varepsilon(t)\|_{H^1(\mathbb{R})}$.

Step 3: Refined estimate of $|\omega(t) - \omega(0)| + |c(t) - c(0)|$. By (3.1), (3.2), and (3.5), we have for any $t \in [0, T^*]$

$$\begin{aligned} \begin{pmatrix} M(u(t)) \\ P(u(t)) \end{pmatrix} &= \begin{pmatrix} M(\varphi_{\omega(t), c(t)}) \\ P(\varphi_{\omega(t), c(t)}) \end{pmatrix} + O\left(\|\varepsilon(t)\|_{H^1(\mathbb{R})}^2\right) \\ &= \begin{pmatrix} M(\varphi_{\omega(0), c(0)}) \\ P(\varphi_{\omega(0), c(0)}) \end{pmatrix} + d''(\omega(0), c(0)) \begin{pmatrix} \omega(t) - \omega(0) \\ c(t) - c(0) \end{pmatrix} + O\left(\|\varepsilon(t)\|_{H^1(\mathbb{R})}^2\right) \\ &\quad + (|\omega(t) - \omega(0)| + |c(t) - c(0)|) \beta(|\omega(t) - \omega(0)| + |c(t) - c(0)|), \end{aligned}$$

which, together with (3.4), the mass and momentum conservation laws and the non-degenerate conditions $\det d''(\omega^0, c^0) < 0$, implies that for sufficient small δ

$$|\omega(t) - \omega(0)| + |c(t) - c(0)| \leq C \left(\|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 + \|\varepsilon(0)\|_{H^1(\mathbb{R})}^2 \right). \quad (3.15)$$

By (3.5), (3.14) and (3.15), we have

$$\|\varepsilon(t)\|_{H^1(\mathbb{R})} + |\omega(t) - \omega(0)| + |c(t) - c(0)| \leq C \|\varepsilon(0)\|_{H^1(\mathbb{R})}. \quad (3.16)$$

Step 4: Conclusion. Now by (3.4), and (3.16), we have for any $t \in [0, T^*]$

$$\begin{aligned} &\inf_{(x^0, \gamma^0) \in \mathbb{R}^2} \left\| u(t, \cdot) - \varphi_{\omega^0, c^0}(\cdot - x^0) e^{i\gamma^0} \right\|_{H^1(\mathbb{R})} \\ &\leq \left\| u(t, \cdot) - \varphi_{\omega^0, c^0}(\cdot - x(t)) e^{i\gamma(t)} \right\|_{H^1(\mathbb{R})} \\ &\leq \left\| u(t, \cdot) - \varphi_{\omega(t), c(t)}(\cdot - x(t)) e^{i\gamma(t)} \right\|_{H^1(\mathbb{R})} + C(|\omega(t) - \omega^0| + |c(t) - c^0|) \end{aligned}$$

$$\begin{aligned}
&\leq \|\varepsilon(t)\|_{H^1(\mathbb{R})} + C(|\omega(t) - \omega(0)| + |c(t) - c(0)| + |\omega(0) - \omega^0| + |c(0) - c^0|) \\
&\leq C\left(\|\varepsilon(0)\|_{H^1(\mathbb{R})} + |\omega(0) - \omega^0| + |c(0) - c^0|\right) \\
&\leq C C_I \delta.
\end{aligned}$$

If choosing $A_0 \geq 2CC_I$, then for any $t \in [0, T^*]$, we have

$$\inf_{(x^0, \gamma^0) \in \mathbb{R}^2} \left\| u(t, \cdot) - \varphi_{\omega^0, c^0}(\cdot - x^0) e^{i\gamma^0} \right\|_{H^1(\mathbb{R})} \leq \frac{1}{2} A_0 \delta,$$

which contradicts with the assumption $T^* < +\infty$ by the continuity of $u(t)$ in $H^1(\mathbb{R})$. This implies $T^* = +\infty$ and completes the proof of Theorem 1.4.

4. DECOMPOSITION OF THE SOLUTION AROUND THE SUM OF TWO TRAVELING WAVES

From now on, we will consider the stability of the solution around the sum of two traveling waves for (1.2) in the energy space as that of gKdV and NLS equations in [20, 21], we shall extend the structure decomposition of the functions (or solutions) around the single traveling wave as in Lemma 2.3 to the corresponding decomposition. It is noticed that the interaction between two traveling waves is weak since they locate far away from each other.

First, we introduce some notations. Let $(\omega_k^0, c_k^0) \in \mathbb{R}^2$ be such that $0 < c_1^0 < c_2^0$ and $(c_k^0)^2 < 4\omega_k^0$, $k = 1, 2$. Let $\alpha < \alpha_0$ be small enough, and $L > L_0$ be large enough, where α_0, L_0 will be determined later. Now we define the following H^1 -tube which is close to the sum of two traveling waves with weak interaction,

$$\begin{aligned}
&\mathcal{U}(\alpha, L, \omega_1^0, c_1^0, \omega_2^0, c_2^0) \\
&\triangleq \left\{ u \in H^1(\mathbb{R}) : \inf_{\substack{x_2 - x_1 > L \\ \gamma_1, \gamma_2 \in \mathbb{R}}} \left\| u - \sum_{k=1}^2 \varphi_{\omega_k^0, c_k^0}(\cdot - x_k) e^{i\gamma_k} \right\|_{H^1(\mathbb{R})} < \alpha \right\}.
\end{aligned}$$

Next, a similar argument, but more delicate, as we adopted in the proof of Lemma 2.3 gives us the useful geometrical decomposition of the functions in the above H^1 -tube. More precisely, we have

Lemma 4.1. *There exist constants $\alpha_0 > 0$ sufficient small, \tilde{L}_0 large enough and $C_{II} > 0$, such that if $u \in \mathcal{U}(\alpha, L, \omega_1^0, c_1^0, \omega_2^0, c_2^0)$ with $\alpha < \alpha_0$, and $L > \tilde{L}_0$, then there exist unique \mathcal{C}^1 functions*

$$\vec{\mathbf{q}} \triangleq (\omega_1, c_1, x_1, \gamma_1, \omega_2, c_2, x_2, \gamma_2) \in (0, +\infty) \times \mathbb{R}^3 \times (0, +\infty) \times \mathbb{R}^3$$

with $(c_k)^2 < 4\omega_k$, $k = 1, 2$, such that

$$\Re \int R_k(x) \overline{\varepsilon(x)} dx = 0, \quad \Re \int \left(i \partial_x R_k + \frac{1}{2} |R_k|^2 R_k \right) (x) \overline{\varepsilon(x)} dx = 0, \quad (4.1)$$

$$\Re \int \partial_x R_k(x) \overline{\varepsilon(x)} dx = 0, \quad \Re \int i R_k(x) \overline{\varepsilon(x)} dx = 0, \quad (4.2)$$

where $k = 1, 2$ and

$$R_k(x) = R(\omega_k, c_k, x_k, \gamma_k; x) \triangleq \varphi_{\omega_k, c_k}(x - x_k) e^{i\gamma_k}, \quad (4.3)$$

$$\varepsilon(x) = \varepsilon(\vec{\mathbf{q}}, u; x) \triangleq u(x) - \sum_{k=1}^2 R_k(x). \quad (4.4)$$

Moreover, we have

$$\|\varepsilon\|_{H^1(\mathbb{R})} + \sum_{k=1}^2 (|\omega_k - \omega_k^0| + |c_k - c_k^0|) \leq C_{II} \alpha. \quad (4.5)$$

- Remark 4.2.** (1) For $k = 1, 2$, the traveling wave $\varphi_{\omega_k, c_k}(x - x_k) e^{i\gamma_k}$ of (1.2) makes sense only for $(c_k)^2 < 4\omega_k$ or $(c_k)^2 = 4\omega_k$ with $c_k > 0$ in [22], thus the parameter α_0 is dependent of the parameters $(\omega_1^0, c_1^0, \omega_2^0, c_2^0)$.
- (2) The condition “ \tilde{L}_0 large enough” is essential in the stability theory of the sum of multi traveling waves in the energy space. It makes sure that the interaction between all traveling waves is weak, hence the implicit function theorem and perturbation theory can be applied.

Proof of Lemma 4.1. First, by the definition of the H^1 -tube, we know that for any $u \in \mathcal{U}(\alpha, L, \omega_1^0, c_1^0, \omega_2^0, c_2^0)$, there exist x_1^0, γ_1^0, x_2^0 and γ_2^0 with $x_2^0 - x_1^0 > L$ such that

$$\|u - R^0\|_{H^1(\mathbb{R})} < \alpha, \quad (4.6)$$

where $R^0(x) \triangleq \sum_{k=1}^2 R_k^0(x)$, $R_k^0(x) \triangleq R(\omega_k^0, c_k^0, x_k^0, \gamma_k^0; x)$. Denote the open H^1 -ball of center $R^0(x)$ and of radius α by $B(R^0, \alpha)$, then $u \in B(R^0, \alpha)$.

Now let

$$\vec{\mathbf{q}}^0 \triangleq (\omega_1^0, c_1^0, x_1^0, \gamma_1^0, \omega_2^0, c_2^0, x_2^0, \gamma_2^0),$$

and define for $k = 1, 2$

$$\begin{aligned} \varrho_1^k(\vec{\mathbf{q}}, u) &= \Re \int R_k(x) \overline{\varepsilon(\vec{\mathbf{q}}, u; x)} dx, \\ \varrho_2^k(\vec{\mathbf{q}}, u) &= \Re \int \left(i\partial_x R_k + \frac{1}{2} |R_k|^2 R_k \right) (x) \overline{\varepsilon(\vec{\mathbf{q}}, u; x)} dx, \\ \varrho_3^k(\vec{\mathbf{q}}, u) &= \Re \int \partial_x R_k(x) \overline{\varepsilon(\vec{\mathbf{q}}, u; x)} dx, \\ \varrho_4^k(\vec{\mathbf{q}}, u) &= \Re \int iR_k(x) \overline{\varepsilon(\vec{\mathbf{q}}, u; x)} dx, \end{aligned}$$

where $R_k(x)$ and $\varepsilon(\vec{\mathbf{q}}, u; x)$ are defined by (4.3) and (4.4). It is easy to see that

$$\varepsilon(\vec{\mathbf{q}}^0, R^0; x) \equiv 0. \quad (4.7)$$

By (4.7) and the fact that

$$\begin{aligned} \left(\frac{\partial}{\partial \omega_k} \varepsilon \right) (\vec{q}^0, R^0 ; x) &= - \frac{\partial}{\partial \omega_k} R_k \Big|_{\vec{q}=\vec{q}^0}, \\ \left(\frac{\partial}{\partial c_k} \varepsilon \right) (\vec{q}^0, R^0 ; x) &= - \frac{\partial}{\partial c_k} R_k \Big|_{\vec{q}=\vec{q}^0}, \\ \left(\frac{\partial}{\partial x_k} \varepsilon \right) (\vec{q}^0, R^0 ; x) &= - \frac{\partial}{\partial x_k} R_k \Big|_{\vec{q}=\vec{q}^0}, \\ \left(\frac{\partial}{\partial \gamma_k} \varepsilon \right) (\vec{q}^0, R^0 ; x) &= - i R_k \Big|_{\vec{q}=\vec{q}^0}, \end{aligned}$$

we have for $k, k' = 1, 2$

$$\begin{aligned} \frac{\partial \varrho_1^k}{\partial \omega_{k'}} (\vec{q}^0, R^0) &= -\Re \int R_k^0(x) \overline{\frac{\partial}{\partial \omega_{k'}} R_{k'} \Big|_{\vec{q}=\vec{q}^0}}(x) dx, \\ \frac{\partial \varrho_1^k}{\partial c_{k'}} (\vec{q}^0, R^0) &= -\Re \int R_k^0(x) \overline{\frac{\partial}{\partial c_{k'}} R_{k'} \Big|_{\vec{q}=\vec{q}^0}}(x) dx, \\ \frac{\partial \varrho_1^k}{\partial x_{k'}} (\vec{q}^0, R^0) &= -\Re \int R_k^0(x) \overline{\frac{\partial}{\partial x_{k'}} R_{k'} \Big|_{\vec{q}=\vec{q}^0}}(x) dx, \\ \frac{\partial \varrho_1^k}{\partial \gamma_{k'}} (\vec{q}^0, R^0) &= \Re \int R_k^0(x) i \overline{R_{k'} \Big|_{\vec{q}=\vec{q}^0}}(x) dx, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \varrho_2^k}{\partial \omega_{k'}} (\vec{q}^0, R^0) &= -\Re \int \left(i \partial_x R_k^0 + \frac{1}{2} |R_k^0|^2 R_k^0 \right) (x) \overline{\frac{\partial}{\partial \omega_{k'}} R_{k'} \Big|_{\vec{q}=\vec{q}^0}}(x) dx, \\ \frac{\partial \varrho_2^k}{\partial c_{k'}} (\vec{q}^0, R^0) &= -\Re \int \left(i \partial_x R_k^0 + \frac{1}{2} |R_k^0|^2 R_k^0 \right) (x) \overline{\frac{\partial}{\partial c_{k'}} R_{k'} \Big|_{\vec{q}=\vec{q}^0}}(x) dx, \\ \frac{\partial \varrho_2^k}{\partial x_{k'}} (\vec{q}^0, R^0) &= -\Re \int \left(i \partial_x R_k^0 + \frac{1}{2} |R_k^0|^2 R_k^0 \right) (x) \overline{\frac{\partial}{\partial x_{k'}} R_{k'} \Big|_{\vec{q}=\vec{q}^0}}(x) dx, \\ \frac{\partial \varrho_2^k}{\partial \gamma_{k'}} (\vec{q}^0, R^0) &= \Re \int \left(i \partial_x R_k^0 + \frac{1}{2} |R_k^0|^2 R_k^0 \right) (x) i \overline{R_{k'} \Big|_{\vec{q}=\vec{q}^0}}(x) dx, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \varrho_3^k}{\partial \omega_{k'}} (\vec{q}^0, R^0) &= -\Re \int \partial_x R_k^0(x) \overline{\frac{\partial}{\partial \omega_{k'}} R_{k'} \Big|_{\vec{q}=\vec{q}^0}}(x) dx, \\ \frac{\partial \varrho_3^k}{\partial c_{k'}} (\vec{q}^0, R^0) &= -\Re \int \partial_x R_k^0(x) \overline{\frac{\partial}{\partial c_{k'}} R_{k'} \Big|_{\vec{q}=\vec{q}^0}}(x) dx, \\ \frac{\partial \varrho_3^k}{\partial x_{k'}} (\vec{q}^0, R^0) &= -\Re \int \partial_x R_k^0(x) \overline{\frac{\partial}{\partial x_{k'}} R_{k'} \Big|_{\vec{q}=\vec{q}^0}}(x) dx, \\ \frac{\partial \varrho_3^k}{\partial \gamma_{k'}} (\vec{q}^0, R^0) &= \Re \int \partial_x R_k^0(x) i \overline{R_{k'} \Big|_{\vec{q}=\vec{q}^0}}(x) dx, \end{aligned}$$

and

$$\begin{aligned}\frac{\partial \varrho_4^k}{\partial \omega_{k'}}(\vec{q}^0, R^0) &= -\Re \int i R_k^0(x) \overline{\frac{\partial}{\partial \omega_{k'}} R_{k'} \Big|_{\vec{q}=\vec{q}^0}}(x) dx, \\ \frac{\partial \varrho_4^k}{\partial c_{k'}}(\vec{q}^0, R^0) &= -\Re \int i R_k^0(x) \overline{\frac{\partial}{\partial c_{k'}} R_{k'} \Big|_{\vec{q}=\vec{q}^0}}(x) dx, \\ \frac{\partial \varrho_4^k}{\partial x_{k'}}(\vec{q}^0, R^0) &= -\Re \int i R_k^0(x) \overline{\frac{\partial}{\partial x_{k'}} R_{k'} \Big|_{\vec{q}=\vec{q}^0}}(x) dx, \\ \frac{\partial \varrho_4^k}{\partial \gamma_{k'}}(\vec{q}^0, R^0) &= \Re \int i R_k^0(x) i \overline{R_{k'} \Big|_{\vec{q}=\vec{q}^0}}(x) dx.\end{aligned}$$

In order to use the implicit function theorem for $\varrho_j^k(\vec{q}, u)$ around (\vec{q}^0, R^0) with $j = 1, 2, 3, 4$ and $k = 1, 2$, we only need to verify that the determinant of the corresponding Jacobian Ξ is nonzero, where

$$\Xi = \begin{pmatrix} \Xi_{1,1} & \Xi_{1,2} \\ \Xi_{2,1} & \Xi_{2,2} \end{pmatrix}, \quad \Xi_{k,k'} = \left(\begin{array}{cccc} \frac{\partial \varrho_1^k}{\partial \omega_{k'}} & \frac{\partial \varrho_1^k}{\partial c_{k'}} & \frac{\partial \varrho_1^k}{\partial x_{k'}} & \frac{\partial \varrho_1^k}{\partial \gamma_{k'}} \\ \frac{\partial \varrho_2^k}{\partial \omega_{k'}} & \frac{\partial \varrho_2^k}{\partial c_{k'}} & \frac{\partial \varrho_2^k}{\partial x_{k'}} & \frac{\partial \varrho_2^k}{\partial \gamma_{k'}} \\ \frac{\partial \varrho_3^k}{\partial \omega_{k'}} & \frac{\partial \varrho_3^k}{\partial c_{k'}} & \frac{\partial \varrho_3^k}{\partial x_{k'}} & \frac{\partial \varrho_3^k}{\partial \gamma_{k'}} \\ \frac{\partial \varrho_4^k}{\partial \omega_{k'}} & \frac{\partial \varrho_4^k}{\partial c_{k'}} & \frac{\partial \varrho_4^k}{\partial x_{k'}} & \frac{\partial \varrho_4^k}{\partial \gamma_{k'}} \end{array} \right) \Big|_{(\vec{q}, u) = (\vec{q}^0, R^0)}. \quad (4.8)$$

First, by a straight calculation, we have for $k = 1, 2$

$$\Xi_{k,k} = \left(\begin{array}{cccc} -\partial_{\omega_k} M(R_k) & -\partial_{c_k} M(R_k) & 0 & 0 \\ -\partial_{\omega_k} P(R_k) & -\partial_{c_k} P(R_k) & 0 & 0 \\ -\Re \int \partial_x R_k \overline{\partial_{\omega_k} R_k} & -\Re \int \partial_x R_k \overline{\partial_{c_k} R_k} & -\Re \int \partial_x R_k \overline{\partial_{x_k} R_k} & -\Re \int \partial_x R_k \overline{i R_k} \\ -\Re \int i R_k \overline{\partial_{\omega_k} R_k} & -\Re \int i R_k \overline{\partial_{c_k} R_k} & -\Re \int i R_k \overline{\partial_{x_k} R_k} & -\Re \int i R_k \overline{i R_k} \end{array} \right) \Big|_{(\vec{q}, u) = (\vec{q}^0, R^0)},$$

then as in the proof of Lemma 2.3, we have

$$\det \Xi_{k,k} = -\|\partial_x \phi_{\omega_k^0, c_k^0}\|_2^2 \|\phi_{\omega_k^0, c_k^0}\|_2^2 \cdot \det d''(\omega_k^0, c_k^0), \quad \text{for } k = 1, 2. \quad (4.9)$$

Next, we consider $\Xi_{k,k'}$ with $k \neq k'$. Since the center distance between \mathcal{R}_1^0 and \mathcal{R}_2^0 is at least L , we have

$$\begin{aligned}\int |\mathcal{R}_1^0(x) \mathcal{R}_2^0(x)| dx &\leq C \int e^{-\frac{\sqrt{4\omega_1^0 - (c_1^0)^2}}{2}|x-x_1^0|} e^{-\frac{\sqrt{4\omega_2^0 - (c_2^0)^2}}{2}|x-x_2^0|} dx \\ &\leq C e^{-\theta|x_1^0 - x_2^0|} \leq C e^{-\theta L},\end{aligned}$$

where $C = C(\omega_1^0, c_1^0, \omega_2^0, c_2^0)$, $\theta = \frac{1}{4} \min \left\{ \sqrt{4\omega_1^0 - (c_1^0)^2}, \sqrt{4\omega_2^0 - (c_2^0)^2} \right\}$, and \mathcal{R}_k^0 takes one of the expressions $R_k|_{\vec{q}=\vec{q}^0}$, $\partial_x R_k|_{\vec{q}=\vec{q}^0}$, $\frac{\partial}{\partial \omega_k} R_k|_{\vec{q}=\vec{q}^0}$, $\frac{\partial}{\partial c_k} R_k|_{\vec{q}=\vec{q}^0}$, $\frac{\partial}{\partial x_k} R_k|_{\vec{q}=\vec{q}^0}$

and $\frac{\partial}{\partial \gamma_k} R_k \Big|_{\vec{q}=\vec{q}^0}$. It follows that for $k \neq k'$ and $j = 1, 2, 3, 4$

$$\left| \frac{\partial \varrho_j^k}{\partial \omega_{k'}} (\vec{q}^0, R^0) \right| + \left| \frac{\partial \varrho_j^k}{\partial c_{k'}} (\vec{q}^0, R^0) \right| + \left| \frac{\partial \varrho_j^k}{\partial x_{k'}} (\vec{q}^0, R^0) \right| + \left| \frac{\partial \varrho_j^k}{\partial \gamma_{k'}} (\vec{q}^0, R^0) \right| \leq C e^{-\theta L},$$

which means

$$\det \Xi_{k,k'} = O \left(e^{-\theta L} \right), \quad \text{for } k \neq k'. \quad (4.10)$$

Now, inserting (4.9) and (4.10) into (4.8), we have

$$\det \Xi = \prod_{k=1}^2 \left(\|\partial_x \phi_{\omega_k^0, c_k^0}\|_2^2 \|\phi_{\omega_k^0, c_k^0}\|_2^2 \cdot \det d'' (\omega_k^0, c_k^0) \right) + O \left(e^{-\theta L} \right),$$

which implies that there exists $L_0 \triangleq L_0(\omega_1^0, c_1^0, \omega_2^0, c_2^0)$ large enough such that

$$\det \Xi \neq 0 \quad \text{for } L > L_0.$$

At last, the implicit function theorem implies the results for any $u \in B(R^0, \alpha)$ with small α , and the estimate (4.5) with constant C_{II} is independent of parameters $x_k, \gamma_k, k = 1, 2$ of the ball $B(R^0, \alpha)$, provided that $x_2 - x_1 > L_0$. \square

Now we apply the above decomposition of the function to the solution $u(t)$ of (1.2) in $[0, T_0]$, and obtain the corresponding dynamical version. More precisely, we have

Lemma 4.3. *Suppose $u \in \mathcal{C}([0, T_0], H^1(\mathbb{R}))$ is a solution to (1.2) with initial data $u_0 \in \mathcal{U}(\alpha, L, \omega_1^0, c_1^0, \omega_2^0, c_2^0)$, and*

$$u(t) \in \mathcal{U} \left(\alpha, \frac{L}{2}, \omega_1^0, c_1^0, \omega_2^0, c_2^0 \right), \quad \text{for any } t \in (0, T_0],$$

where $\alpha < \alpha_0$ and $\frac{L}{2} > \tilde{L}_0$ with α_0 and \tilde{L}_0 given by Lemma 4.1. Then there exist unique \mathcal{C}^1 functions

$$\vec{q}(t) \triangleq (\omega_1(t), c_1(t), x_1(t), \gamma_1(t), \omega_2(t), c_2(t), x_2(t), \gamma_2(t))$$

on $[0, T_0]$ with values $(0, +\infty) \times \mathbb{R}^3 \times (0, +\infty) \times \mathbb{R}^3$ and $(c_k(t))^2 < 4\omega_k(t), k = 1, 2$ for all $t \in [0, T_0]$ such that

$$\Re \int R_k(t) \overline{\varepsilon(t)} dx = 0, \quad \Re \int \left(i \partial_x R_k + \frac{1}{2} |R_k|^2 R_k \right) (t) \overline{\varepsilon(t)} dx = 0, \quad (4.11)$$

$$\Re \int \partial_x R_k(t) \overline{\varepsilon(t)} dx = 0, \quad \Re \int i R_k(t) \overline{\varepsilon(t)} dx = 0, \quad (4.12)$$

where $k = 1, 2$ and

$$R_k(t, x) = R(\omega_k(t), c_k(t), x_k(t), \gamma_k(t); x) \triangleq \varphi_{\omega_k(t), c_k(t)}(x - x_k(t)) e^{i\gamma_k(t)}, \quad (4.13)$$

$$\varepsilon(t, x) = \varepsilon(\vec{q}(t), u(t, x); x) \triangleq u(t, x) - \sum_{k=1}^2 R_k(t, x). \quad (4.14)$$

Moreover, for any $t \in [0, T_0]$, we have

$$\|\varepsilon(t)\|_{H^1(\mathbb{R})} + \sum_{k=1}^2 (|\omega_k(t) - \omega_k^0| + |c_k(t) - c_k^0|) \leq C_{II} \alpha, \quad (4.15)$$

$$x_2(t) - x_1(t) \geq \frac{L}{2}, \quad (4.16)$$

and

$$\begin{aligned} & |\dot{\omega}_k(t)| + |\dot{c}_k(t)| + |\dot{x}_k(t) - c_k(t)| + |\dot{\gamma}_k(t) - \omega_k(t)| \\ & \leq C_{II} \left(\|\varepsilon(t)\|_{H^1} + e^{-\theta_1(\frac{L}{2} + \theta_1 t)} \right), \end{aligned} \quad (4.17)$$

$$\text{where } \theta_1 = \frac{1}{4} \min \left\{ \sqrt{4\omega_1^0 - (c_1^0)^2}, \sqrt{4\omega_2^0 - (c_2^0)^2}, c_2^0 - c_1^0 \right\}.$$

Proof. Applying Lemma 4.1 to $u(t)$ for all $t \in [0, T_0]$, we can obtain (4.11), (4.12) and (4.15). It remains to show (4.16) and (4.17).

Since $u(t) \in \mathcal{U}(\alpha, \frac{L}{2}, \omega_1^0, c_1^0, \omega_2^0, c_2^0)$ for $t \in (0, T_0]$, there exist $x_1^0(t)$, $x_2^0(t)$, $\gamma_1^0(t)$ and $\gamma_2^0(t)$ such that $x_2^0(t) - x_1^0(t) \geq \frac{L}{2}$ and

$$\left\| u(t, \cdot) - \sum_{k=1}^2 R_k^0(t, \cdot) \right\|_{H^1(\mathbb{R})} < \alpha, \quad (4.18)$$

where

$$R_k^0(t, x) = R(\omega_k^0, c_k^0, x_k^0(t), \gamma_k^0(t); x) = \varphi_{\omega_k^0, c_k^0}(x - x_k^0(t)) e^{i\gamma_k^0(t)}.$$

As in the proof of Lemma 4.1, we know that for $k = 1, 2$,

$$|x_k(t) - x_k^0(t)| < C_{II} \alpha,$$

which implies

$$x_2(t) - x_1(t) \geq \frac{L}{4},$$

for sufficient small α and sufficient large L . We will improve this estimate by the dynamics of $\dot{x}_k(t)$ and $x_2(0) - x_1(0) \geq \frac{L}{2}$.

The \mathcal{C}^1 regularity of $\omega_k(t)$, $c_k(t)$, $x_k(t)$ and $\gamma_k(t)$ in t can be shown by a standard regularization argument, we can refer to [19] for more details. Now we formally verify (4.17) by the equation of $\varepsilon(t)$, and the orthogonal structure (4.11) and (4.12). A simple calculation gives that

$$\begin{aligned} i\partial_t \varepsilon + \mathcal{L}\varepsilon = & \sum_{k=1}^2 \left(-i\partial_t R_k - \partial_x^2 R_k - \frac{1}{2}i|R_k|^2 \partial_x R_k + \frac{1}{2}iR_k^2 \overline{\partial_x R_k} - \frac{3}{16}|R_k|^4 R_k \right) \\ & - \frac{1}{2}i|R_1 + R_2|^2 \partial_x (R_1 + R_2) + \frac{1}{2}i|R_1|^2 \partial_x R_1 + \frac{1}{2}i|R_2|^2 \partial_x R_2 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2}i(R_1 + R_2)^2 \overline{\partial_x(R_1 + R_2)} - \frac{1}{2}iR_1^2 \overline{\partial_x R_1} - \frac{1}{2}iR_2^2 \overline{\partial_x R_2} \\
& - \frac{3}{16}|R_1 + R_2|^4(R_1 + R_2) + \frac{3}{16}|R_1|^4 R_1 + \frac{3}{16}|R_2|^4 R_2 \\
& + H.O.T,
\end{aligned} \tag{4.19}$$

where $\mathcal{L}\varepsilon$ and $H.O.T$ are defined by

$$\begin{aligned}
\mathcal{L}\varepsilon = & \partial_x^2 \varepsilon + \frac{1}{2}i \left[|R_1 + R_2|^2 \partial_x \varepsilon + 2\partial_x(R_1 + R_2) \Re \left(\overline{(R_1 + R_2)} \varepsilon \right) \right] \\
& - \frac{1}{2}i \left[(R_1 + R_2)^2 \overline{\partial_x \varepsilon} + 2\overline{\partial_x(R_1 + R_2)} (R_1 + R_2) \varepsilon \right] \\
& + \frac{3}{16} \left[|R_1 + R_2|^4 \varepsilon + 4|R_1 + R_2|^2 \Re \left(\overline{(R_1 + R_2)} \varepsilon \right) \right],
\end{aligned}$$

and

$$\begin{aligned}
H.O.T = & -\frac{1}{2}i \left[2\partial_x \varepsilon \Re \left(\overline{(R_1 + R_2)} \varepsilon \right) + \partial_x(R_1 + R_2) |\varepsilon|^2 + |\varepsilon|^2 \partial_x \varepsilon \right] \\
& + \frac{1}{2}i \left[2\overline{\partial_x \varepsilon} (R_1 + R_2) \varepsilon + \overline{\partial_x(R_1 + R_2)} \varepsilon^2 + \varepsilon^2 \overline{\partial_x \varepsilon} \right] \\
& - \frac{3}{16} \left[4(R_1 + R_2) \left(\Re \left(\overline{(R_1 + R_2)} \varepsilon \right) \right)^2 + 2|R_1 + R_2|^2 (R_1 + R_2) |\varepsilon|^2 \right. \\
& \quad + 4|R_1 + R_2|^2 \varepsilon \Re \left(\overline{(R_1 + R_2)} \varepsilon \right) + 4(R_1 + R_2) |\varepsilon|^2 \Re \left(\overline{(R_1 + R_2)} \varepsilon \right) \\
& \quad + 4 \left(\Re \left(\overline{(R_1 + R_2)} \varepsilon \right) \right)^2 \varepsilon + 2|R_1 + R_2|^2 |\varepsilon|^2 \varepsilon + (R_1 + R_2) |\varepsilon|^4 \\
& \quad \left. + 4 \Re \left(\overline{(R_1 + R_2)} \varepsilon \right) |\varepsilon|^2 \varepsilon + |\varepsilon|^4 \varepsilon \right].
\end{aligned}$$

By (1.8), we know that $R_k(t, x)$, $k = 1, 2$, satisfy

$$\begin{aligned}
& -i\partial_t R_k - \partial_x^2 R_k - \frac{1}{2}i|R_k|^2 \partial_x R_k + \frac{1}{2}iR_k^2 \overline{\partial_x R_k} - \frac{3}{16}|R_k|^4 R_k \\
= & -i\dot{\omega}_k(t) \frac{\partial}{\partial \omega_k} R_k(t) - i\dot{c}_k(t) \frac{\partial}{\partial c_k} R_k(t) \\
& -i(\dot{x}_k(t) - c_k(t)) \frac{\partial}{\partial x_k} R_k(t) - i(\dot{\gamma}_k(t) - \omega_k(t)) \frac{\partial}{\partial \gamma_k} R_k(t).
\end{aligned}$$

Inserting the above identity into (4.19), we have

$$i\partial_t \varepsilon + \mathcal{L}\varepsilon + \sum_{k=1}^2 \left(i\dot{\omega}_k(t) \frac{\partial}{\partial \omega_k} R_k(t) + i\dot{c}_k(t) \frac{\partial}{\partial c_k} R_k(t) \right) \tag{4.20}$$

$$\begin{aligned}
& + \sum_{k=1}^2 \left(i(\dot{x}_k(t) - c_k(t)) \frac{\partial}{\partial x_k} R_k(t) + i(\dot{\gamma}_k(t) - \omega_k(t)) \frac{\partial}{\partial \gamma_k} R_k(t) \right) \\
= & -\frac{1}{2}i|R_1 + R_2|^2 \partial_x(R_1 + R_2) + \frac{1}{2}i|R_1|^2 \partial_x R_1 + \frac{1}{2}i|R_2|^2 \partial_x R_2
\end{aligned} \tag{4.21}$$

$$+ \frac{1}{2}i(R_1 + R_2)^2 \overline{\partial_x(R_1 + R_2)} - \frac{1}{2}iR_1^2 \overline{\partial_x R_1} - \frac{1}{2}iR_2^2 \overline{\partial_x R_2} \tag{4.22}$$

$$\begin{aligned}
& -\frac{3}{16}|R_1 + R_2|^4(R_1 + R_2) + \frac{3}{16}|R_1|^4 R_1 + \frac{3}{16}|R_2|^4 R_2 \\
& + H.O.T.
\end{aligned} \tag{4.23}$$

Because $R_1(t, x)$ and $R_2(t, x)$ have exponential decay and their centers locate far away from each other (distance is at least $L/4$ from (4.16)), we know that (4.21)-(4.23) are weak interaction terms between $R_1(t, x)$ and $R_2(t, x)$.

Now we will combine the above equation about $\varepsilon(t, x)$ with the orthogonal structures (4.11), (4.12) for $k = 1, 2$ to show (4.17). On one hand, multiplying the above equation by $\overline{R_k(t)}$, $i\partial_x R_k(t) + \frac{1}{2}|R_k|^2 R_k(t)$, $\partial_x R_k(t)$, $i R_k(t)$, for $k = 1, 2$, and taking the imaginary part, we have from (4.15) and $\det \Xi_{k,k} > 0$, $k = 1, 2$ that

$$|\dot{\omega}_k(t)| + |\dot{c}_k(t)| + |\dot{x}_k(t) - c_k(t)| + |\dot{\gamma}_k(t) - \omega_k(t)| \leq C \left(\|\varepsilon(t)\|_{H^1} + e^{-\theta_1 \frac{L}{4}} \right), \tag{4.24}$$

where $\|\varepsilon(t)\|_{H^1}$ comes from the contribution of the linear terms $i\partial_t \varepsilon + \mathcal{L}\varepsilon$ and $H.O.T$ term, and $e^{-\theta_1 \frac{L}{4}}$ comes from the weak interaction terms in (4.21)-(4.23), especially the fact that $x_2(t) - x_1(t) \geq \frac{L}{4}$. On the other hand, in order to get the precise estimate, we estimate $x_2(t) - x_1(t)$ as following

$$\begin{aligned}
\dot{x}_2(t) - \dot{x}_1(t) &= (\dot{x}_2(t) - c_2(t)) - (\dot{x}_1(t) - c_1(t)) + (c_2(t) - c_1(t)) \\
&\geq (c_2(t) - c_1(t)) - 2C \left(C_{II}\alpha + e^{-2\theta_1 \frac{L}{4}} \right) \\
&\geq (c_2^0 - c_1^0) - 2C_{II}\alpha - 2C \left(C_{II}\alpha + e^{-2\theta_1 \frac{L}{4}} \right) \\
&\geq \frac{c_2^0 - c_1^0}{4} \geq \theta_1
\end{aligned}$$

for sufficient small α and large L . This yields

$$x_2(t) - x_1(t) = x_2(0) - x_1(0) + \int_0^t (\dot{x}_2(s) - \dot{x}_1(s)) \, ds \geq \frac{L}{2} + \theta_1 t,$$

from which we can obtain (4.16) and (4.17). \square

5. MONOTONICITY FORMULAS FOR THE DERIVATIVE NLS

As shown in Section 3, under the non-degenerate condition

$$\det d''(\omega, c) < 0,$$

there are two key points to show the stability of the solution of (1.2) around the single traveling wave in the energy space besides of the modulation analysis. One is the action functional $\mathfrak{J}_{\omega(0), c(0)}(u(t))$, which is a conserved quantity, and is used to obtain the refined estimate of $\|\varepsilon(t)\|_{H^1(\mathbb{R})}$ by the perturbation argument,

$$\|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \leq C \|\varepsilon(0)\|_{H^1(\mathbb{R})}^2 + C \left(|\omega(t) - \omega(0)|^2 + |c(t) - c(0)|^2 \right).$$

The other is the conservation laws of mass and momentum³, which is enough to show the refined estimate that

$$|\omega(t) - \omega(0)| + |c(t) - c(0)| \leq C \left(\|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 + \|\varepsilon(0)\|_{H^1(\mathbb{R})}^2 \right).$$

As for the multi-traveling waves case, inspired by the ideas in [21], we will introduce the localized action functional $\mathfrak{E}(u(t)) = E(u(t)) + \mathfrak{F}(t)$, which is almost conserved and can be used to obtain the refined estimate of $\|\varepsilon(t)\|_{H^1(\mathbb{R})}$ by the perturbation argument. Since it is not enough to refine the estimates of $|\omega_k(t) - \omega_k(0)| + |c_k(t) - c_k(0)|$, $k = 1, 2$, only by the conservation laws of mass and momentum, we need to introduce some kinds of the localized functionals $\mathfrak{Q}_{\pm,0}(t)$ and $\mathfrak{Q}_{0,\pm}(t)$ and characterize its dynamical estimate, which is related to $|\omega_k(t) - \omega_k(0)| + |c_k(t) - c_k(0)|$. They are the main goals in this section.

5.1. Monotone result for the line $x = \bar{x}^0 + \sigma t$. First of all, we introduce a suitable cutoff function h , which is a nondecreasing \mathcal{C}^3 function with

$$h(x) = \begin{cases} 0, & \text{if } x < -1, \\ 1, & \text{if } x > 1, \end{cases}$$

$$(h'(x))^2 \leq Ch(x), \quad (h''(x))^2 \leq Ch'(x), \quad \text{for } x \in \mathbb{R}, \quad (5.1)$$

and

$$h'(x) > 0, \quad \text{for } x \in (-1, 1).$$

Next, by setting⁴

$$\bar{x}^0 = \frac{x_1^0 + x_2^0}{2}, \quad \sigma = 2 \frac{\omega_2(0) - \omega_1(0)}{c_2(0) - c_1(0)}, \quad a = \frac{L^2}{64},$$

and

$$\mathfrak{h}(t, x) = h\left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t + a}}\right), \quad \mathfrak{g}(t, x) = 1 - \mathfrak{h}(t, x), \quad (5.2)$$

we define the functional $\mathfrak{F}(t)$, including the localized mass and momentum about each traveling wave,

$$\begin{aligned} \mathfrak{F}(t) = & \omega_1(0) \int \frac{1}{2} |u(t)|^2 \mathfrak{g}(t) \, dx + c_1(0) \int \left(-\frac{1}{2} \Im(\bar{u} \partial_x u) + \frac{1}{8} |u|^4 \right) (t) \mathfrak{g}(t) \, dx \\ & + \omega_2(0) \int \frac{1}{2} |u(t)|^2 \mathfrak{h}(t) \, dx + c_2(0) \int \left(-\frac{1}{2} \Im(\bar{u} \partial_x u) + \frac{1}{8} |u|^4 \right) (t) \mathfrak{h}(t) \, dx \end{aligned} \quad (5.3)$$

³Of course, $\omega(t)$ and $c(t)$ are related to the mass and momentum of the corresponding traveling wave at time t .

⁴By the assumption (c) in Theorem 1.6, we know that $c_1^0 < \sigma < c_2^0$.

A simple calculation gives us that

$$\begin{aligned}
\mathfrak{F}(t) &= \omega_1(0) \int \frac{1}{2} |u(t)|^2 dx + c_1(0) \int \left(-\frac{1}{2} \Im(\bar{u} \partial_x u)(t) + \frac{1}{8} |u(t)|^4 \right) dx \\
&\quad + (\omega_2(0) - \omega_1(0)) \int \frac{1}{2} |u(t)|^2 \mathfrak{h}(t) dx \\
&\quad + (c_2(0) - c_1(0)) \int \left(-\frac{1}{2} \Im(\bar{u} \partial_x u) + \frac{1}{8} |u|^4 \right)(t) \mathfrak{h}(t) dx \\
&= \omega_1(0) M(u(t)) + c_1(0) P(u(t)) + \mathfrak{Q}(t),
\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}
\mathfrak{Q}(t) &= (\omega_2(0) - \omega_1(0)) \int \frac{1}{2} |u(t)|^2 \mathfrak{h}(t) dx \\
&\quad + (c_2(0) - c_1(0)) \int \left(-\frac{1}{2} \Im(\bar{u} \partial_x u) + \frac{1}{8} |u|^4 \right)(t) \mathfrak{h}(t) dx.
\end{aligned}$$

Next, we have

Lemma 5.1. *Let $u(t)$ be a solution of (1.2) satisfying the assumption of Lemma 4.3 on $[0, T_0]$. Then there exist $C > 0$ such that*

$$\frac{d}{dt} \mathfrak{Q}(t) \leq \frac{C}{(t+a)^{3/2}} \left(\int_{|x-\bar{x}^0-\sigma t| < \sqrt{t+a}} |u(t, x)|^2 dx + \left(\int_{|x-\bar{x}^0-\sigma t| < \sqrt{t+a}} |u(t, x)|^2 dx \right)^3 \right).$$

Proof. By (1.2), we have

$$\begin{aligned}
&\frac{1}{c_2(0) - c_1(0)} \frac{d}{dt} \mathfrak{Q}(t) \\
&= \frac{\sigma}{2\sqrt{t+a}} \int \Im(\bar{u}(t) \partial_x u(t)) h' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) dx \\
&\quad + \frac{1}{4} \sigma \int |u(t)|^2 h' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) \left[-\frac{\sigma}{\sqrt{t+a}} - \frac{x - \bar{x}^0 - \sigma t}{2(t+a)^{3/2}} \right] dx \\
&\quad - \frac{1}{\sqrt{t+a}} \int |\partial_x u(t)|^2 h' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) dx \\
&\quad + \frac{1}{4(t+a)^{3/2}} \int |u(t)|^2 h''' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) dx \\
&\quad + \frac{1}{16\sqrt{t+a}} \int |u(t)|^6 h' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) dx \\
&\quad - \frac{1}{2} \int \Im(\bar{u}(t) \partial_x u(t)) h' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) \left[-\frac{\sigma}{\sqrt{t+a}} - \frac{x - \bar{x}^0 - \sigma t}{2(t+a)^{3/2}} \right] dx \\
&\quad + \frac{1}{8} \int |u(t)|^4 h' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) \left[-\frac{\sigma}{\sqrt{t+a}} - \frac{x - \bar{x}^0 - \sigma t}{2(t+a)^{3/2}} \right] dx.
\end{aligned}$$

Now by introducing

$$v(t, x) \triangleq e^{-i\frac{1}{2}\sigma x} u(t, x),$$

we have

$$\frac{1}{c_2(0) - c_1(0)} \frac{d}{dt} \mathfrak{Q}(t) \quad (5.5)$$

$$= -\frac{1}{\sqrt{t+a}} \int |\partial_x v(t)|^2 h' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) dx \\ + \frac{1}{4(t+a)} \int \Im \left(\overline{v(t)} \partial_x v(t) \right) h' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) \frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} dx \quad (5.6)$$

$$+ \frac{1}{4(t+a)^{3/2}} \int |v(t)|^2 h''' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) dx \\ + \frac{1}{16\sqrt{t+a}} \int |v(t)|^6 h' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) dx \quad (5.7)$$

$$+ \frac{1}{8} \int |v(t)|^4 h' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) \left[-\frac{\sigma}{\sqrt{t+a}} - \frac{x - \bar{x}^0 - \sigma t}{2(t+a)^{3/2}} \right] dx. \quad (5.8)$$

Estimate of (5.6). By the Cauchy-Schwarz inequality and the support property of h' , we have

$$\left| \frac{1}{4(t+a)} \int \Im \left(\overline{v(t)} \partial_x v(t) \right) h' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) \frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} dx \right| \quad (5.9) \\ \leq \frac{1}{8\sqrt{t+a}} \int |\partial_x v(t)|^2 h' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) dx + \frac{C}{(t+a)^{3/2}} \int_{|x - \bar{x}^0 - \sigma t| < \sqrt{t+a}} |v(t)|^2 dx.$$

Estimate of (5.7). By (5.1), Lemma 3.3 in [21], we have

$$\frac{1}{16\sqrt{t+a}} \int |v(t)|^6 h' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) dx \quad (5.10) \\ \leq \frac{1}{8\sqrt{t+a}} \int |\partial_x v(t)|^2 h' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) dx + \frac{C}{(t+a)^{3/2}} \left(\int_{|x - \bar{x}^0 - \sigma t| < \sqrt{t+a}} |v(t)|^2 dx \right)^3.$$

Before we estimate the fourth order term, we first give the following useful lemma, which is similar to Lemma 3.3 in [21] for the sixth order term.

Lemma 5.2. *Let $h(x) \geq 0$ be a \mathcal{C}^2 bounded function. Then for any $w \in H^1$, we have*

$$\int |w(x)|^4 h(x) dx \leq C \left(\int |w_x|^2 h dx \right)^{1/2} \left(\int_{\text{supp } h} |w|^2 dx \right)^{3/2} + \int |w|^2 h_x dx \int_{\text{supp } h} |w|^2 dx,$$

where $\text{supp } h$ denotes the support of h .

Proof. First, the Leibnitz rule gives us that

$$\frac{d}{dx} (|w|^2 h) = 2h\Re(\bar{w}w_x) + |w|^2 h_x,$$

therefore, it follows from $w \in H^1(\mathbb{R})$ and the boundedness of h that

$$\left(|w|^2 h\right)(x) = 2 \int_{-\infty}^x h \Re(\bar{w} w_x) \, dx + \int_{-\infty}^x |w|^2 h_x \, dx,$$

which implies that

$$\begin{aligned} \left\| |w|^2 h \right\|_{L^\infty} &\leq 2 \int |w| |w_x| h \, dx + \int |w|^2 |h_x| \, dx \\ &\leq 2 \left(\int |w_x|^2 h \, dx \right)^{1/2} \left(\int |w|^2 h \, dx \right)^{1/2} + \int |w|^2 |h_x| \, dx. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \int |w|^4 h \, dx &\leq \left\| |w|^2 h \right\|_{L^\infty} \int_{\text{supp } h} |w|^2 \, dx \\ &\leq \left(2 \left(\int |w_x|^2 h \, dx \right)^{1/2} \left(\int_{\text{supp } h} |w|^2 h \, dx \right)^{1/2} + \int |w|^2 h_x \, dx \right) \cdot \int_{\text{supp } h} |w|^2 \, dx \\ &\leq C \left(\int |w_x|^2 h \, dx \right)^{1/2} \left(\int_{\text{supp } h} |w|^2 \, dx \right)^{3/2} + \int |w|^2 h_x \, dx \int_{\text{supp } h} |w|^2 \, dx. \end{aligned}$$

This completes the proof. \square

Estimate of (5.8). First, by the fact that $h' \geq 0$ and $\sigma > 0$, we have⁵

$$-\frac{1}{8} \int |v(t)|^4 h' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) \frac{\sigma}{\sqrt{t+a}} \, dx \leq 0. \quad (5.11)$$

Second, by the boundedness of h' and h'' , and the fact that $\text{supp } h' \subset (-1, 1)$, $h' \geq 0$, Lemma 5.2 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\left| -\frac{1}{8} \int |v(t)|^4 h' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) \frac{x - \bar{x}^0 - \sigma t}{2(t+a)^{3/2}} \, dx \right| \quad (5.12) \\ &\leq \frac{1}{16(t+a)} \int |v(t)|^4 h' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) \, dx \\ &\leq \frac{C}{t+a} \left(\int |\partial_x v(t)|^2 h' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) \, dx \right)^{1/2} \left(\int_{|x - \bar{x}^0 - \sigma t| < \sqrt{t+a}} |v(t)|^2 \, dx \right)^{3/2} \\ &\quad + \frac{1}{16(t+a)^{3/2}} \left(\int_{|x - \bar{x}^0 - \sigma t| < \sqrt{t+a}} |v(t)|^2 \, dx \right)^2 \\ &\leq \frac{1}{8\sqrt{t+a}} \int |\partial_x v(t)|^2 h' \left(\frac{x - \bar{x}^0 - \sigma t}{\sqrt{t+a}} \right) \, dx \\ &\quad + \frac{C}{(t+a)^{3/2}} \left(\int_{|x - \bar{x}^0 - \sigma t| < \sqrt{t+a}} |v(t)|^2 \, dx \right)^2 + \frac{C}{(t+a)^{3/2}} \left(\int_{|x - \bar{x}^0 - \sigma t| < \sqrt{t+a}} |v(t)|^2 \, dx \right)^3. \end{aligned}$$

⁵Here we throw away the first term for $\sigma > 0$ because of $c_2^0 > c_1^0 > 0$ and $\omega_2^0 > \omega_1^0$. In general case, we cannot use the estimate in Lemma 5.2 since the resulted decay in t is just $(t+a)^{-1}$, which is non-integral(critical).

Inserting (5.9), (5.10), (5.11) and (5.12) into (5.5), we can obtain the result by the fact that $|v(t)| = |u(t)|$. \square

Next, we turn to compute the local mass of the solution $u(t, x)$ to (1.2) in the region $|x - \bar{x}^0 - \sigma t| < \sqrt{t+a}$.

Lemma 5.3. *Let $u(t)$ be a solution of (1.2) satisfying the assumption of Lemma 4.3 on $[0, T_0]$. Then there exist $C, L_2, \alpha_2 > 0$ such that if $L \geq L_2$ and $0 < \alpha < \alpha_2$ we have*

$$\int_{|x - \bar{x}^0 - \sigma t| < \sqrt{t+a}} |u(t, x)|^2 dx \leq 2 \int |\varepsilon(t, x)|^2 dx + C e^{-\theta_2(L+\theta_2 t)} < 1 \quad (5.13)$$

for $t \in [0, T_0]$, where

$$0 < \theta_2 \triangleq \frac{1}{16} \min \left(\sigma - c_1^0, c_2^0 - \sigma, \frac{\sqrt{4\omega_1^0 - (c_1^0)^2}}{4}, \frac{\sqrt{4\omega_2^0 - (c_2^0)^2}}{4} \right) < \frac{\theta_1}{4}.$$

Proof. By (4.15), the second inequality in (5.13) is obvious for sufficient small α and large L . We now show the first inequality in (5.13). It follows from Lemma 4.3 that

$$u(t, x) = \sum_{k=1}^2 R_k(t, x) + \varepsilon(t, x),$$

where $R_k(t, x) = \varphi_{\omega_k(t), c_k(t)}(x - x_k(t)) e^{i\gamma_k(t)}$.

First, by the Sobolev inequality and (4.15), we have

$$\|\varepsilon(t)\|_{L^\infty(\mathbb{R})} \leq C \|\varepsilon(t)\|_{H^1(\mathbb{R})} \leq C C_{II} \alpha.$$

This yields $\|\varepsilon(t)\|_{L^\infty} < \frac{1}{2}$ for α is small enough.

Next, we estimate the contribution of the traveling waves. When x is in the region $|x - \bar{x}^0 - \sigma t| < \sqrt{t+a} \leq \sqrt{t} + \frac{L}{8}$, we have

$$\begin{aligned} |x - x_1(t)| &= |(x - \bar{x}^0 - \sigma t) - (x_1(t) - \bar{x}^0 - \sigma t)| \\ &\geq |x_1(t) - \bar{x}^0 - \sigma t| - |x - \bar{x}^0 - \sigma t| \\ &\geq |x_1(t) - \bar{x}^0 - \sigma t| - \sqrt{t} - \frac{L}{8}. \end{aligned} \quad (5.14)$$

By (4.15) and (4.17), we have for sufficient small α and sufficient large L that

$$\begin{aligned} \frac{d}{dt} (\bar{x}^0 + \sigma t - x_1(t)) &= \sigma - (\dot{x}_1(t) - c_1(t)) - c_1(t) \\ &\geq \sigma - C_{II} \left(C_{II} \alpha + e^{-\theta_1(\frac{L}{2} + \theta_1 t)} \right) - c_1^0 - C_{II} \alpha \\ &\geq \frac{\sigma - c_1^0}{2}, \end{aligned}$$

and so,

$$\bar{x}^0 + \sigma t - x_1(t) \geq \bar{x}^0 - x_1(0) + \frac{\sigma - c_1^0}{2} t \geq \frac{L}{4} + \frac{\sigma - c_1^0}{2} t.$$

Now inserting the above estimate into (5.14), we obtain for α small enough and L large enough that

$$\begin{aligned} |x - x_1(t)| &\geq \bar{x}^0 + \sigma t - x_1(t) - \sqrt{t} - \frac{L}{8} \\ &\geq \frac{L}{16} + \frac{\sigma - c_1^0}{4} t + \left(\frac{\sigma - c_1^0}{4} t - \sqrt{t} + \frac{L}{16} \right) \\ &\geq \frac{L}{16} + \frac{\sigma - c_1^0}{4} t \geq \frac{L}{16} + 4\theta_2 t, \end{aligned} \quad (5.15)$$

which implies that

$$|R_1(t, x)| \leq C e^{-\frac{\sqrt{4\omega_1(t) - c_1^2(t)}}{2} |x - x_1(t)|} \leq C e^{-\frac{\sqrt{4\omega_1^0 - (c_1^0)^2}}{4} (\frac{L}{16} + 4\theta_2 t)} \leq C e^{-16\theta_2 (\frac{L}{16} + 4\theta_2 t)}.$$

In a similar way, we have

$$|R_2(t, x)| \leq C e^{-16\theta_2 (\frac{L}{16} + 4\theta_2 t)}.$$

Thus, we have

$$\int_{|x - \bar{x}^0 - \sigma t| < \sqrt{t+a}} |u(t, x)|^2 dx \leq 2 \int_{|x - \bar{x}^0 - \sigma t| < \sqrt{t+a}} |\varepsilon(t, x)|^2 dx + C e^{-\theta_2(L + \theta_2 t)}.$$

This completes the proof. \square

As a consequence of (4.15), Lemma 5.1 and Lemma 5.3, we have

Proposition 5.4. *Let $u(t)$ be a solution of (1.2) satisfying the assumptions of Lemma 4.3 on $[0, T_0]$, then there exist $C, L_2, \alpha_2 > 0$ such that if $L \geq L_2$ and $0 < \alpha < \alpha_2$ we have*

$$\mathfrak{Q}(t) - \mathfrak{Q}(0) \leq \frac{C}{L} \sup_{0 \leq s < t} \int |\varepsilon(s, x)|^2 dx + C e^{-\theta_2 L} \quad \text{for } t \in [0, T_0].$$

5.2. Monotone results for the lines $x = \bar{x}^0 + \sigma_{0,\pm}t$ and $x = \bar{x}^0 + \sigma_{\pm,0}t$. In order to get the refined estimates of $|\omega_k(t) - \omega_k(0)|$ and $|c_k(t) - c_k(0)|$ for $k = 1, 2$ in next section, we introduce the following monotone functionals for the different lines and characterize its property here. First we denote⁶

$$\sigma_{+,0} = \sigma_{0,-} = \frac{\sigma + c_2^0}{2}, \quad \sigma_{-,0} = \sigma_{0,+} = \frac{\sigma + \max\{c_1^0, 0\}}{2}, \quad (5.16)$$

and define

$$\mathfrak{h}_{\pm,0}(t, x) = h\left(\frac{x - \bar{x}^0 - \sigma_{\pm,0}t}{\sqrt{t+a}}\right), \quad \mathfrak{h}_{0,\pm}(t, x) = h\left(\frac{x - \bar{x}^0 - \sigma_{0,\pm}t}{\sqrt{t+a}}\right),$$

and

$$\begin{aligned} \frac{1}{c_2(0) - c_1(0)} \mathfrak{Q}_{+,0}(t) &= \frac{\sigma_{+,0}}{2} \int \frac{1}{2} |u(t, x)|^2 \mathfrak{h}_{+,0}(t, x) dx \\ &\quad + \int \left(-\frac{1}{2} \Im \left(\overline{u(t, x)} \partial_x u(t, x) \right) + \frac{1}{8} |u(t, x)|^4 \right) \mathfrak{h}_{+,0}(t, x) dx, \end{aligned}$$

⁶It is easy to verify that $c_1^0 < \sigma_{\pm,0}$, $\sigma_{0,\pm} < c_2^0$ by assumption (c) in Theorem 1.6.

$$\begin{aligned}
\frac{1}{c_2(0) - c_1(0)} \mathfrak{Q}_{-,0}(t) &= \frac{\sigma_{-,0}}{2} \int \frac{1}{2} |u(t,x)|^2 \mathfrak{h}_{-,0}(t,x) \, dx \\
&\quad + \int \left(-\frac{1}{2} \Im \left(\overline{u(t,x)} \partial_x u(t,x) \right) + \frac{1}{8} |u(t,x)|^4 \right) \mathfrak{h}_{-,0}(t,x) \, dx, \\
\frac{1}{\omega_2(0) - \omega_1(0)} \mathfrak{Q}_{0,+}(t) &= \int \frac{1}{2} |u(t,x)|^2 \mathfrak{h}_{0,+}(t,x) \, dx \\
&\quad + \frac{2}{\sigma_{0,+}} \int \left(-\frac{1}{2} \Im \left(\overline{u(t,x)} \partial_x u(t,x) \right) + \frac{1}{8} |u(t,x)|^4 \right) \mathfrak{h}_{0,+}(t,x) \, dx,
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\omega_2(0) - \omega_1(0)} \mathfrak{Q}_{0,-}(t) &= \int \frac{1}{2} |u(t,x)|^2 \mathfrak{h}_{0,-}(t,x) \, dx \\
&\quad + \frac{2}{\sigma_{0,-}} \int \left(-\frac{1}{2} \Im \left(\overline{u(t,x)} \partial_x u(t,x) \right) + \frac{1}{8} |u(t,x)|^4 \right) \mathfrak{h}_{0,-}(t,x) \, dx.
\end{aligned}$$

By the similar arguments as that in the proof of Proposition 5.4, we have

Corollary 5.5. *Let $u(t)$ be a solution of (1.2) satisfying the assumption of Lemma 4.3 on $[0, T_0]$. Then there exist $C, L_3, \alpha_3 > 0$ such that if $L \geq L_3$ and $0 < \alpha < \alpha_3$ we have for $t \in [0, T_0]$,*

$$\begin{aligned}
\mathfrak{Q}_{\pm,0}(t) - \mathfrak{Q}_{\pm,0}(0) &\leq \frac{C}{L} \sup_{0 < s < t} \int |\varepsilon(s,x)|^2 \, dx + C e^{-\theta_3 L}, \\
\mathfrak{Q}_{0,\pm}(t) - \mathfrak{Q}_{0,\pm}(0) &\leq \frac{C}{L} \sup_{0 < s < t} \int |\varepsilon(s,x)|^2 \, dx + C e^{-\theta_3 L},
\end{aligned}$$

where

$$0 < \theta_3 = \frac{1}{16} \min \left(\sigma_{\pm,0} - c_1^0, c_2^0 - \sigma_{\pm,0}, \frac{\sqrt{4\omega_1^0 - (c_1^0)^2}}{4}, \frac{\sqrt{4\omega_2^0 - (c_2^0)^2}}{4} \right) \leq \theta_2.$$

Remark 5.6. The introduction of the constant “a” aims to ensure that the variations of $\mathfrak{Q}(t)$, $\mathfrak{Q}_{\pm,0}(t)$ and $\mathfrak{Q}_{0,\pm}(t)$ in Proposition 5.4 and Corollary 5.5 can be small by comparing with the L^2 estimate of the remainder term $\varepsilon(t)$ in the interval $[0, T_0]$, up to small errors $e^{-\theta_2 L}$ and $e^{-\theta_3 L}$.

6. STABILITY OF THE SUM OF TWO TRAVELING WAVES

In this section, we show the result in Theorem 1.6. Let ω_k^0 and c_k^0 satisfy the assumptions in Theorem 1.6. Fix

$$\theta_0 = \min(\theta_1, \theta_2, \theta_3).$$

Let α_0 be defined by Lemma 4.1 and $A_0 > 2$, $\delta_0 = \delta_0(A_0)$, $L_0 = L_0(A_0)$ be chosen later. Suppose that $u(t)$ is the solution of (1.2) with initial data $u_0 \in \mathcal{U}(\delta, L, \omega_1^0, c_1^0, \omega_2^0, c_2^0)$.

Then there exist $x_k^0 \in \mathbb{R}$ and $\gamma_k^0 \in \mathbb{R}$, $k = 1, 2$ such that

$$\left\| u_0(\cdot) - \sum_{k=1}^2 \varphi_{\omega_k^0, c_k^0}(\cdot - x_k^0) e^{i\gamma_k^0} \right\|_{H^1(\mathbb{R})} < \delta,$$

where $\delta < \alpha_0$ and $x_2^0 - x_1^0 > L$. Now we define

$$T^* = \sup \left\{ t \geq 0, \sup_{\tau \in [0, t]} \inf_{\substack{x_2^0 - x_1^0 > \frac{L}{2} \\ \gamma_k^0 \in \mathbb{R}, k=1,2}} \left\| u(\tau, \cdot) - \sum_{k=1}^2 \varphi_{\omega_k^0, c_k^0}(\cdot - x_k^0) e^{i\gamma_k^0} \right\|_{H^1(\mathbb{R})} \leq A_0 \left(\delta + e^{-\theta_0 \frac{L}{2}} \right) \right\}.$$

By the continuity of $u(t)$ in H^1 , we know that $T^* > 0$. In order to prove Theorem 1.6, it suffices to show $T^* = +\infty$ for some $A_0 > 2$, $\delta_0 > 0$, and L_0 .

We argue with contradiction. Suppose that $T^* < +\infty$, we know that for any $t \in [0, T^*]$, there exist $(x_k^0(t), \gamma_k^0(t)) \in \mathbb{R}^2$, $k = 1, 2$ such that $x_2^0(t) \geq x_1^0(t) + \frac{L}{2}$ and

$$\left\| u(t, \cdot) - \sum_{k=1}^2 \varphi_{\omega_k^0, c_k^0}(\cdot - x_k^0(t)) e^{i\gamma_k^0(t)} \right\|_{H^1(\mathbb{R})} \leq A_0 \left(\delta + e^{-\theta_0 \frac{L}{2}} \right).$$

Step 1: Decomposition of the solution $u(t)$ around the sum of two traveling waves.

Let $\tilde{L}_0 > 0$ be determined by Lemma 4.1, and L_2, L_3 be determined by Proposition 5.4 and Corollary 5.5, and choose $\delta_0 > 0$ small enough and L_0 large enough, such that for $\delta < \delta_0$ and $L > L_0(A_0) > \max \{2\tilde{L}_0, L_2, L_3\}$,

$$A_0 \left(\delta + e^{-\theta_0 \frac{L}{2}} \right) < \alpha_0.$$

Now by Lemma 4.3, there exists a unique \mathcal{C}^1 functions

$$\vec{q}(t) = (\omega_1(t), c_1(t), x_1(t), \gamma_1(t), \omega_2(t), c_2(t), x_2(t), \gamma_2(t))$$

on $[0, T^*]$ with $(c_k(t))^2 < 4\omega_k(t)$, $k = 1, 2$ such that the remainder term

$$\varepsilon(t, x) = u(t, x) - \sum_{k=1}^2 R_k(t, x) \tag{6.1}$$

has the orthogonality conditions (4.11)-(4.12) on $[0, T^*]$, for $k = 1, 2$, where $R_k(t, x) \triangleq \varphi_{\omega_k(t), c_k(t)}(x - x_k(t)) e^{i\gamma_k(t)}$. Moreover, we have

$$\|\varepsilon(t)\|_{H^1(\mathbb{R})} + \sum_{k=1}^2 (|\omega_k(t) - \omega_k^0| + |c_k(t) - c_k^0|) \leq C_{II} A_0 \left(\delta + e^{-\theta_0 \frac{L}{2}} \right), \tag{6.2}$$

$$|x_2(t) - x_1(t)| > \frac{L}{2} + \theta_0 t. \tag{6.3}$$

for any $t \in [0, T^*]$, and

$$\|\varepsilon(0)\|_{H^1(\mathbb{R})} + \sum_{k=1}^2 (|\omega_k(0) - \omega_k^0| + |c_k(0) - c_k^0|) \leq C_{II} \delta. \tag{6.4}$$

If necessary, we can take δ_0 sufficiently small and L_0 sufficiently large to ensure that $C_{II}A_0 \left(\delta + e^{-\theta_0 \frac{L}{2}} \right) < 1$.

Step 2: Action functional and refined estimate of $\varepsilon(t)$ and $|\mathfrak{Q}(t) - \mathfrak{Q}(0)|$. For this purpose, we introduce the following localized action functional,

$$\mathfrak{E}(u(t)) = E(u(t)) + \mathfrak{F}(t), \quad (6.5)$$

where $\mathfrak{F}(t)$ is defined by (5.4), which is not conserved along the flow (1.2). Nevertheless from Proposition 5.4, we know that it is almost conserved or controllable. First, by the linearized argument and the orthogonal structures (4.11) and (4.12), we have

Lemma 6.1. *For $t \in [0, T^*]$, we have the following expansion formula.*

$$\begin{aligned} \mathfrak{E}(u(t)) &= \sum_{k=1}^2 \mathfrak{J}_{\omega_k(0), c_k(0)}(R_k(0)) + \mathfrak{H}(\varepsilon(t), \varepsilon(t)) \\ &\quad + O\left(\sum_{k=1}^2 \left(|\omega_k(t) - \omega_k(0)|^2 + |c_k(t) - c_k(0)|^2\right)\right) \\ &\quad + \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \beta\left(\|\varepsilon(t)\|_{H^1(\mathbb{R})}\right) + O\left(e^{-8\theta_0(\frac{L}{2} + 8\theta_0 t)}\right) \end{aligned}$$

where

$$\begin{aligned} &\mathfrak{H}(\varepsilon(t), \varepsilon(t)) \\ &= \frac{1}{2} \int |\partial_x \varepsilon(t)|^2 dx - \frac{3}{32} \sum_{k=1}^2 \int |R_k(t)|^4 |\varepsilon(t)|^2 dx - \frac{3}{8} \sum_{k=1}^2 \int |R_k(t)|^2 [\Re(\overline{R_k} \varepsilon)(t)]^2 dx \\ &\quad + \frac{\omega_1(t)}{2} \int |\varepsilon(t)|^2 \mathfrak{g}(t) dx + \frac{\omega_2(t)}{2} \int |\varepsilon(t)|^2 \mathfrak{h}(t) dx \\ &\quad - \frac{c_1(t)}{2} \Im \int (\overline{\varepsilon} \partial_x \varepsilon)(t) \mathfrak{g}(t) dx - \frac{c_2(t)}{2} \Im \int (\overline{\varepsilon} \partial_x \varepsilon)(t) \mathfrak{h}(t) dx \\ &\quad + \sum_{k=1}^2 \frac{c_k(t)}{4} \int |R_k(t)|^2 |\varepsilon(t)|^2 dx + \sum_{k=1}^2 \frac{c_k(t)}{2} \int [\Re(\overline{R_k} \varepsilon)(t)]^2 dx. \end{aligned}$$

Proof. See the proof in Appendix B. □

On one hand, by Lemma 6.1, we have

$$\begin{aligned} \mathfrak{E}(u(0)) &= \sum_{k=1}^2 \mathfrak{J}_{\omega_k(0), c_k(0)}(R_k(0)) + \mathfrak{H}(\varepsilon(0), \varepsilon(0)) \\ &\quad + \|\varepsilon(0)\|_{H^1(\mathbb{R})}^2 \beta\left(\|\varepsilon(0)\|_{H^1(\mathbb{R})}\right) + O\left(e^{-8\theta_0 \frac{L}{2}}\right), \end{aligned} \quad (6.6)$$

and

$$\mathfrak{E}(u(t)) = \sum_{k=1}^2 \mathfrak{J}_{\omega_k(0), c_k(0)}(R_k(0)) + \mathfrak{H}(\varepsilon(t), \varepsilon(t))$$

$$\begin{aligned}
& + O \left(\sum_{k=1}^2 \left(|\omega_k(t) - \omega_k(0)|^2 + |c_k(t) - c_k(0)|^2 \right) \right) \\
& + \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \beta \left(\|\varepsilon(t)\|_{H^1(\mathbb{R})} \right) + O \left(e^{-8\theta_0(\frac{L}{2} + 8\theta_0 t)} \right). \tag{6.7}
\end{aligned}$$

On the other hand, by (5.4), the conservation laws of mass, momentum and energy, we have

$$\mathfrak{E}(u(t)) - \mathfrak{E}(u(0)) = \mathfrak{Q}(t) - \mathfrak{Q}(0). \tag{6.8}$$

Combining (6.6), (6.7) and (6.8), we have

$$\begin{aligned}
& \mathfrak{H}(\varepsilon(t), \varepsilon(t)) + \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \beta \left(\|\varepsilon(t)\|_{H^1(\mathbb{R})} \right) \\
& \leq \mathfrak{Q}(t) - \mathfrak{Q}(0) + \mathfrak{H}(\varepsilon(0), \varepsilon(0)) + \|\varepsilon(0)\|_{H^1(\mathbb{R})}^2 \beta \left(\|\varepsilon(0)\|_{H^1(\mathbb{R})} \right) \\
& \quad + O \left(\sum_{k=1}^2 \left(|\omega_k(t) - \omega_k(0)|^2 + |c_k(t) - c_k(0)|^2 \right) \right) + O \left(e^{-4\theta_0 L} \right) \\
& \leq \mathfrak{Q}(t) - \mathfrak{Q}(0) + C \|\varepsilon(0)\|_{H^1(\mathbb{R})}^2 + O \left(e^{-4\theta_0 L} \right) \\
& \quad + O \left(\sum_{k=1}^2 \left(|\omega_k(t) - \omega_k(0)|^2 + |c_k(t) - c_k(0)|^2 \right) \right), \tag{6.9}
\end{aligned}$$

where we use (6.4) and the fact that $\mathfrak{H}(\varepsilon(0), \varepsilon(0)) \leq C \|\varepsilon(0)\|_{H^1(\mathbb{R})}^2$ in the second inequality.

Now by the orthogonal structures (4.11), (4.12) and the standard localized argument, we have

Lemma 6.2. *There exists $C_1 > 0$ such that*

$$\mathfrak{H}(\varepsilon(t), \varepsilon(t)) \geq C_1 \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2.$$

Proof. See the proof in Appendix C. □

Now by (6.2), (6.9) and the above lemma, we have

$$\begin{aligned}
\frac{C_1}{2} \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 & \leq \mathfrak{Q}(t) - \mathfrak{Q}(0) + C \|\varepsilon(0)\|_{H^1(\mathbb{R})}^2 + C e^{-4\theta_0 L} \\
& \quad + O \left(\sum_{k=1}^2 \left(|\omega_k(t) - \omega_k(0)|^2 + |c_k(t) - c_k(0)|^2 \right) \right). \tag{6.10}
\end{aligned}$$

By Proposition 5.4, we have

$$\begin{aligned}
\frac{C_1}{2} \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 & \leq \frac{C}{L} \sup_{0 < s < t} \int |\varepsilon(s)|^2 + C \|\varepsilon(0)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L} \\
& \quad + O \left(\sum_{k=1}^2 \left(|\omega_k(t) - \omega_k(0)|^2 + |c_k(t) - c_k(0)|^2 \right) \right), \tag{6.11}
\end{aligned}$$

and

$$|\mathfrak{Q}(t) - \mathfrak{Q}(0)| \leq \frac{C}{L} \sup_{0 < s < t} \int |\varepsilon(s)|^2 + C \|\varepsilon(0)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L}$$

$$+ O \left(\sum_{k=1}^2 \left(|\omega_k(t) - \omega_k(0)|^2 + |c_k(t) - c_k(0)|^2 \right) \right). \quad (6.12)$$

Step 3: Refined estimates of $|c_k(t) - c_k(0)|$ and $|\omega_k(t) - \omega_k(0)|$.

By the definitions of $\mathfrak{Q}_{+,0}(t)$ and $\mathfrak{Q}(t)$, we have

$$\begin{aligned} \frac{\mathfrak{Q}_{+,0}(t) - \mathfrak{Q}(t)}{c_2(0) - c_1(0)} &= \frac{\sigma_{+,0} - \sigma}{2} \int \frac{1}{2} |u(t, x)|^2 \mathfrak{h}_{+,0}(t, x) \, dx \\ &\quad + \frac{\sigma}{2} \int \frac{1}{2} |u(t, x)|^2 [\mathfrak{h}_{+,0}(t, x) - \mathfrak{h}(t, x)] \, dx \\ &\quad + \int \left(-\frac{1}{2} \Im(\bar{u} \partial_x u) + \frac{1}{8} |u|^4 \right) (t, x) [\mathfrak{h}_{+,0}(t, x) - \mathfrak{h}(t, x)] \, dx \end{aligned} \quad (6.13)$$

On one hand, by (5.15), we know that $\mathfrak{h}_{+,0} - \mathfrak{h} \equiv 0$ for $x < x_1(t) + L/16$ or $x > x_2(t) - L/16$, which implies that

$$\int (|R(t, x)| + |\partial_x R(t, x)|) |\mathfrak{h}_{+,0}(t, x) - \mathfrak{h}(t, x)| \, dx \leq C e^{-\theta_0 L}.$$

This together with the Cauchy-Schwarz inequality yields

$$\left| \int \left(|u(t)|^2 + \left| \Im(\bar{u}(t) \partial_x u(t)) \right| + |u(t)|^4 \right) [\mathfrak{h}_{+,0}(t) - \mathfrak{h}(t)] \, dx \right| \leq C \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L}. \quad (6.14)$$

On the other hand, it follows from (4.11) that

$$\left| \int |u(t, x)|^2 \mathfrak{h}_{+,0}(t, x) \, dx - \int |R_2(t, x)|^2 \, dx \right| \leq C \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L}. \quad (6.15)$$

Thus, Inserting (6.14) and (6.15) into (6.13), we have

$$\left| \frac{\mathfrak{Q}_{+,0}(t) - \mathfrak{Q}(t)}{c_2(0) - c_1(0)} - \frac{\sigma_{+,0} - \sigma}{2} M(R_2(t)) \right| \leq C \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L}.$$

Particularly, we have for $t > 0$ and $t = 0$ that

$$- \left(\frac{\mathfrak{Q}_{+,0}(t) - \mathfrak{Q}(t)}{c_2(0) - c_1(0)} - \frac{\sigma_{+,0} - \sigma}{2} M(R_2(t)) \right) \leq C \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L},$$

and

$$\frac{\mathfrak{Q}_{+,0}(0) - \mathfrak{Q}(0)}{c_2(0) - c_1(0)} - \frac{\sigma_{+,0} - \sigma}{2} M(R_2(0)) \leq C \|\varepsilon(0)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L}.$$

Summing up the above two inequalities, we obtain from (6.12) and Corollary 5.5 that

$$\begin{aligned} &\frac{\sigma_{+,0} - \sigma}{2} (M(R_2(t)) - M(R_2(0))) \\ &\leq \frac{\mathfrak{Q}_{+,0}(t) - \mathfrak{Q}_{+,0}(0)}{c_2(0) - c_1(0)} - \frac{\mathfrak{Q}(t) - \mathfrak{Q}(0)}{c_2(0) - c_1(0)} + C \sup_{s \in [0, t]} \|\varepsilon(s)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L} \\ &\leq \frac{\mathfrak{Q}_{+,0}(t) - \mathfrak{Q}_{+,0}(0)}{c_2(0) - c_1(0)} + \left| \frac{\mathfrak{Q}(t) - \mathfrak{Q}(0)}{c_2(0) - c_1(0)} \right| + C \sup_{s \in [0, t]} \|\varepsilon(s)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L} \end{aligned}$$

$$\leq C \sup_{s \in [0, t]} \|\varepsilon(s)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L}. \quad (6.16)$$

By the similar arguments as above, we can show that

$$\left| \frac{\mathfrak{Q}(t) - \mathfrak{Q}_{-,0}(t)}{c_2(0) - c_1(0)} - \frac{\sigma - \sigma_{-,0}}{2} M(R_2(t)) \right| \leq C \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L},$$

and

$$-\frac{\sigma - \sigma_{-,0}}{2} (M(R_2(t)) - M(R_2(0))) \leq C \sup_{s \in [0, t]} \|\varepsilon(s)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L}. \quad (6.17)$$

Combining (6.16) with (6.17), we obtain

$$|M(R_2(t)) - M(R_2(0))| \leq C \sup_{s \in [0, t]} \|\varepsilon(s)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L}. \quad (6.18)$$

By the similar arguments as above and the definitions of $\mathfrak{Q}_{0,\pm}(t)$ and $\mathfrak{Q}(t)$, we can show that

$$|P(R_2(t)) - P(R_2(0))| \leq C \sup_{s \in [0, t]} \|\varepsilon(s)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L}. \quad (6.19)$$

Now by the mass conservation and the orthogonality condition (4.11), we have

$$\left| \sum_{k=1}^2 (M(R_k(t)) - M(R_k(0))) \right| \leq C \sup_{s \in [0, t]} \|\varepsilon(s)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L}, \quad (6.20)$$

where we used the fact that $|x_2(t) - x_1(t)| > \frac{L}{2} + \theta_0 t$. Thus, by (6.18) and (6.20), we obtain

$$|M(R_1(t)) - M(R_1(0))| \leq C \sup_{s \in [0, t]} \|\varepsilon(s)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L}. \quad (6.21)$$

In a similar way, we have

$$|P(R_1(t)) - P(R_1(0))| \leq C \sup_{s \in [0, t]} \|\varepsilon(s)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L}. \quad (6.22)$$

By (4.15) and the non-degenerate condition

$$\det d''(\omega_k^0, c_k^0) < 0$$

for $k = 1, 2$. We can now refine the estimates of $|\omega_k(t) - \omega_k(0)| + |c_k(t) - c_k(0)|$.

Lemma 6.3. *For any $t \in [0, T^*]$, we have for $k = 1, 2$*

$$|\omega_k(t) - \omega_k(0)| + |c_k(t) - c_k(0)| \leq C \sup_{s \in [0, t]} \|\varepsilon(s)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L}.$$

Proof. On one hand, by (6.2) and

$$\det d''(\omega_k^0, c_k^0) < 0,$$

we have for sufficiently small α_0 that

$$2 \det d''(\omega_k^0, c_k^0) < \det d''(\omega_k(t), c_k(t)) < \frac{1}{2} \det d''(\omega_k^0, c_k^0) < 0. \quad (6.23)$$

On the other hand, by the Taylor formula, we have

$$\begin{aligned} & \begin{pmatrix} M(R_k(t)) - M(R_k(0)) \\ P(R_k(t)) - P(R_k(0)) \end{pmatrix} = \begin{pmatrix} M(\varphi_{\omega_k(t), c_k(t)}) - M(\varphi_{\omega_k(0), c_k(0)}) \\ P(\varphi_{\omega_k(t), c_k(t)}) - P(\varphi_{\omega_k(0), c_k(0)}) \end{pmatrix} \\ & = d''(\omega_k(0), c_k(0)) \begin{pmatrix} \omega_k(t) - \omega_k(0) \\ c_k(t) - c_k(0) \end{pmatrix} + O\left(|\omega_k(t) - \omega_k(0)|^2 + |c_k(t) - c_k(0)|^2\right). \end{aligned} \quad (6.24)$$

By (6.18), (6.19), (6.21), (6.22), (6.23) and (6.24), we can obtain the result. \square

Step 4: Conclusion. Combining (6.11) and Lemma 6.3, we have for any $t \in [0, T^*]$

$$\begin{aligned} \frac{C_1}{2} \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 & \leq \frac{C}{L} \sup_{s \in [0, t]} \|\varepsilon(s)\|_{H^1(\mathbb{R})}^2 + C \sup_{s \in [0, t]} \left(\beta(\|\varepsilon(t)\|_{H^1(\mathbb{R})}) \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \right) \\ & \quad + C \|\varepsilon(0)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L}. \end{aligned}$$

By (4.15), and taking α_0 sufficiently small and L_0 sufficiently large, we have for any $t \in [0, T^*]$

$$\|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \leq C \|\varepsilon(0)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L}.$$

This together Lemma 6.3 implies that

$$\|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 + \sum_{k=1}^2 (|\omega_k(t) - \omega_k(0)| + |c_k(t) - c_k(0)|) \leq C \|\varepsilon(0)\|_{H^1(\mathbb{R})}^2 + C e^{-\theta_0 L}.$$

Last, we have

$$\begin{aligned} & \inf_{x_2^0 - x_1^0 > \frac{L}{2}, \gamma_k^0 \in \mathbb{R}} \left\| u(t, \cdot) - \sum_{k=1}^2 \varphi_{\omega_k^0, c_k^0}(\cdot - x_k^0) e^{i\gamma_k^0} \right\|_{H^1(\mathbb{R})} \\ & \leq \left\| u(t, \cdot) - \sum_{k=1}^2 \varphi_{\omega_k^0, c_k^0}(\cdot - x_k(t)) e^{i\gamma_k(t)} \right\|_{H^1(\mathbb{R})} \\ & \leq \left\| u(t, \cdot) - \sum_{k=1}^2 \varphi_{\omega_k(t), c_k(t)}(\cdot - x_k(t)) e^{i\gamma_k(t)} \right\|_{H^1(\mathbb{R})} + C \left(\sum_{k=1}^2 (|\omega_k(t) - \omega_k^0| + |c_k(t) - c_k^0|) \right) \\ & \leq \|\varepsilon(t)\|_{H^1(\mathbb{R})} + C \left(\sum_{k=1}^2 (|\omega_k(t) - \omega_k(0)| + |c_k(t) - c_k(0)| + |\omega_k(0) - \omega_k^0| + |c_k(0) - c_k^0|) \right) \\ & \leq C \left(\|\varepsilon(0)\|_{H^1(\mathbb{R})} + \sum_{k=1}^2 (|\omega_k(0) - \omega_k^0| + |c_k(0) - c_k^0|) \right) + C e^{-\theta_0 \frac{L}{2}} \\ & \leq C C_{II} \left(\delta + e^{-\theta_0 \frac{L}{2}} \right). \end{aligned}$$

If choosing $A_0 \geq 2CC_{II}$, then for any $t \in [0, T^*]$, we have

$$\inf_{x_2^0 - x_1^0 > \frac{L}{2}, \gamma_k^0 \in \mathbb{R}} \left\| u(t, \cdot) - \sum_{k=1}^2 \varphi_{\omega_k^0, c_k^0}(\cdot - x_k^0) e^{i\gamma_k^0} \right\|_{H^1(\mathbb{R})} \leq \frac{1}{2} A_0 \delta,$$

which contradicts with the assumption $T^* < +\infty$ by the continuity of $u(t)$ in $H^1(\mathbb{R})$. This implies $T^* = +\infty$ and completes the proof of Theorem 1.6.

A. THE COERCIVITY OF THE QUADRATIC TERM

In this appendix, we prove Proposition 2.1. The proof of Part (1) is the same as that in Proposition 2.8 (a) [37]. As for Part (2), the proof is divided into several steps.

Step 1: Spectral decomposition. First of all, it follows from the exponential decay of $\phi_{\omega, c}$ that \mathcal{L}_+ is a relatively compact perturbation of the operator $-\frac{1}{2}\partial_x^2 + \frac{1}{2}\left(\omega - \frac{c^2}{4}\right)$. By Weyl's theorem in [30], we obtain that the essential spectrum of \mathcal{L}_+ on $L^2(\mathbb{R})$ is

$$\sigma_{\text{ess}}(\mathcal{L}_+) = \sigma_{\text{ess}}\left(-\frac{1}{2}\partial_x^2 + \frac{1}{2}\left(\omega - \frac{c^2}{4}\right)\right) = \left[\frac{1}{2}\left(\omega - \frac{c^2}{4}\right), +\infty\right).$$

Moreover, all spectrum below the lower bound of the essential spectrum are either an isolated point of $\sigma(\mathcal{L}_+)$ or an eigenvalue of finite multiplicity of \mathcal{L}_+ .

Next, since $\phi_{\omega, c}$ satisfies

$$\left(\omega - \frac{c^2}{4}\right) \phi_{\omega, c} - \partial_x^2 \phi_{\omega, c} - \frac{3}{16} \phi_{\omega, c}^5 = -\frac{c}{2} \phi_{\omega, c}^3, \quad (\text{A.1})$$

then by differentiating equation (A.1) with respect to x , we obtain

$$\mathcal{L}_+ \partial_x \phi_{\omega, c} = 0. \quad (\text{A.2})$$

Therefore, by $\partial_x \phi_{\omega, c} \in L^2(\mathbb{R})$, we obtain from (A.2) that 0 is an eigenvalue of \mathcal{L}_+ . By a classical ODE argument as in [37], we obtain

$$\ker \mathcal{L}_+ = \text{span} \{ \partial_x \phi_{\omega, c} \}. \quad (\text{A.3})$$

Thus, it follows from Sturm-Liouville theory that 0 is the second eigenvalue of \mathcal{L}_+ , and moreover \mathcal{L}_+ enjoys only one negative eigenvalue $-\lambda_1^2$ with a $L^2(\mathbb{R})$ normalized eigenfunction χ . More precisely, we have

$$\mathcal{L}_+ \chi = -\lambda_1^2 \chi \quad \text{with} \quad \|\chi\|_2 = 1. \quad (\text{A.4})$$

Now, define

$$\mu \triangleq \inf \left\{ \frac{(\mathcal{L}_+ \psi, \psi)}{(\psi, \psi)} : \psi \in L^2(\mathbb{R}), (\psi, \chi) = (\psi, \partial_x \phi_{\omega, c}) = 0 \right\}, \quad (\text{A.5})$$

then by a classical variational argument, it is easy to see that $\mu > 0$. Therefore, the space $L^2(\mathbb{R})$ can be decomposed as a direct sum as follows

$$L^2 = N \bigoplus \ker \mathcal{L}_+ \bigoplus P, \quad (\text{A.6})$$

where $N = \text{span} \{ \chi \}$, $\ker \mathcal{L}_+$ is defined by (A.3), and P is a closed subspace of L^2 such that

$$(\mathcal{L}_+ \psi, \psi) \geq \mu(\psi, \psi), \quad \text{for any } \psi \in P. \quad (\text{A.7})$$

Step 2: Nonnegative property. We show

$$\inf \left\{ \frac{(\mathcal{L}_+ \psi, \psi)}{(\psi, \psi)} : \psi \in L^2, (\psi, \phi_{\omega,c}) = (\psi, \phi_{\omega,c}^3) = (\psi, \partial_x \phi_{\omega,c}) = 0 \right\} \geq 0.$$

In fact, by differentiating equation (A.1) with respect to c and ω , we have

$$\mathcal{L}_+ \partial_c \phi_{\omega,c} = \frac{c}{2} \phi_{\omega,c} - \frac{1}{2} \phi_{\omega,c}^3, \quad \mathcal{L}_+ \partial_\omega \phi_{\omega,c} = -\phi_{\omega,c}. \quad (\text{A.8})$$

On one hand, (A.6) allows us to decompose $\partial_c \phi_{\omega,c}$ and $\partial_\omega \phi_{\omega,c}$ as follows,

$$\partial_c \phi_{\omega,c} = a_1 \chi + b_1 \partial_x \phi_{\omega,c} + p_1, \quad \partial_\omega \phi_{\omega,c} = a_2 \chi + b_2 \partial_x \phi_{\omega,c} + p_2. \quad (\text{A.9})$$

where a_1, a_2, b_1 and b_2 are constants; χ is defined by (A.4); p_1 and p_2 belong to the subspace P defined by (A.7). On the other hand, for any $\psi \in L^2(\mathbb{R})$ with

$$(\psi, \phi_{\omega,c}) = (\psi, \phi_{\omega,c}^3) = (\psi, \partial_x \phi_{\omega,c}) = 0,$$

we decompose ψ as follows

$$\psi = a\chi + p, \quad \text{with } a \in \mathbb{R}, \text{ and } p \in P. \quad (\text{A.10})$$

By some straight calculations, we have

$$(\mathcal{L}_+ \psi, \psi) = -\lambda_1^2 a^2 + (\mathcal{L}_+ p, p). \quad (\text{A.11})$$

Then, it follows from $(\psi, \phi_{\omega,c}) = 0$ and (A.8) that

$$(\psi, \mathcal{L}_+ \partial_\omega \phi_{\omega,c}) = 0,$$

which together with (A.9) and (A.10) implies that

$$-aa_2 \lambda_1^2 + (\mathcal{L}_+ p, p_2) = 0. \quad (\text{A.12})$$

By a similar argument as above, we have

$$-aa_1 \lambda_1^2 + (\mathcal{L}_+ p, p_1) = 0. \quad (\text{A.13})$$

Next, since

$$\det d''(\omega, c) < 0,$$

there exists $(\xi_1, \xi_2) \in \mathbb{R}^2$ such that

$$\begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix} d''(\omega, c) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} > 0. \quad (\text{A.14})$$

Now, let $(\xi_1, \xi_2) \in \mathbb{R}^2$ satisfy (A.14), and

$$p_0 \triangleq \xi_1 p_1 + \xi_2 p_2,$$

then by a straight calculation, we have

$$\begin{aligned} (\mathcal{L}_+ p_0, p_0) &= \xi_1^2 (\mathcal{L}_+ p_1, p_1) + 2\xi_1 \xi_2 (\mathcal{L}_+ p_1, p_2) + \xi_2^2 (\mathcal{L}_+ p_2, p_2) \\ &= \xi_1^2 (\mathcal{L}_+ \partial_c \phi_{\omega, c}, \partial_c \phi_{\omega, c}) + 2\xi_1 \xi_2 (\mathcal{L}_+ \partial_c \phi_{\omega, c}, \partial_\omega \phi_{\omega, c}) \\ &\quad + \xi_2^2 (\mathcal{L}_+ \partial_\omega \phi_{\omega, c}, \partial_\omega \phi_{\omega, c}) + \xi_1^2 a_1^2 \lambda_1^2 + 2\xi_1 \xi_2 a_1 a_2 \lambda_1^2 + \xi_2^2 a_2^2 \lambda_1^2 \\ &= - \begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix} d''(\omega, c) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + (a_1 \xi_1 + a_2 \xi_2)^2 \lambda_1^2 \\ &< (a_1 \xi_1 + a_2 \xi_2)^2 \lambda_1^2. \end{aligned} \tag{A.15}$$

Next, by (A.13), (A.12), (A.15) and the Cauchy-Schwarz inequality, it is easy to see that,

$$(\mathcal{L}_+ p, p) \geq \frac{(\mathcal{L}_+ p, p_0)^2}{(\mathcal{L}_+ p_0, p_0)} \geq \frac{a^2 \lambda_1^2 (a_1 \xi_1 + a_2 \xi_2)^2}{(a_1 \xi_1 + a_2 \xi_2)^2} = a^2 \lambda_1^2,$$

which, together (A.11), implies that,

$$(\mathcal{L}_+ \psi, \psi) \geq 0.$$

Step 3: Positive property. Last we show

$$\inf \left\{ \frac{(\mathcal{L}_+ \psi, \psi)}{(\psi, \psi)} : \psi \in L^2(\mathbb{R}), (\psi, \phi_{\omega, c}) = (\psi, \phi_{\omega, c}^3) = (\psi, \partial_x \phi_{\omega, c}) = 0 \right\} > 0.$$

We argue by contradiction. Suppose that there exists a sequence $\psi_n \in L^2(\mathbb{R})$ such that

$$(\mathcal{L}_+ \psi_n, \psi_n) \rightarrow 0,$$

with

$$(\psi_n, \phi_{\omega, c}) = (\psi_n, \phi_{\omega, c}^3) = (\psi_n, \partial_x \phi_{\omega, c}) = 0 \text{ and } (\psi_n, \psi_n) = 1.$$

By a decomposition similar as (A.10), we have for any n

$$\psi_n = a_n \chi + p_n, \quad \text{with } a_n \in \mathbb{R}, \text{ and } p_n \in P,$$

moreover, $(\mathcal{L}_+ p_n, p_0) = (a_1 \xi_1 + a_2 \xi_2) a_n \lambda_1^2$. Therefore, by the similar arguments as in Step 2, we have

$$\begin{aligned} 0 \leftarrow (\mathcal{L}_+ \psi_n, \psi_n) &\geq -a_n^2 \lambda_1^2 + \frac{a_n^2 (\xi_1 a_1 + \xi_2 a_2)^2 \lambda_1^4}{(\mathcal{L}_+ p_0, p_0)} \\ &= a_n^2 \lambda_1^2 \left(\frac{\lambda_1^2 (\xi_1 a_1 + \xi_2 a_2)^2}{(\mathcal{L}_+ p_0, p_0)} - 1 \right). \end{aligned}$$

Thus, it follows from (A.15) that,

$$a_n \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

which implies that

$$(\mathcal{L}_+ p_n, p_n) \rightarrow 0.$$

Thus $p_n \rightarrow 0$ in $L^2(\mathbb{R})$, which is in contradiction with $(\psi_n, \psi_n) = 1$. This ends the proof.

B. THE LINEARIZATION OF THE ACTION FUNCTIONAL

In this part, we show Lemma 6.1. First of all, we show the following claim,

Claim 1. Let \mathcal{R}_k be one of the expression R_k , $\partial_x R_k$ and $\partial_x^2 R_k$, and \mathfrak{g} and \mathfrak{h} be defined by (5.2), then

$$\int |\mathcal{R}_1(t, x) \mathcal{R}_2(t, x)| \, dx \leq C e^{-8\theta_2(\frac{L}{2} + 8\theta_2 t)}. \quad (\text{B.1})$$

$$\int |\mathcal{R}_1(t, x) \mathfrak{h}(t, x)| \, dx + \int |\mathcal{R}_2(t, x) \mathfrak{g}(t, x)| \, dx \leq C e^{-8\theta_2(\frac{L}{2} + 8\theta_2 t)}, \quad (\text{B.2})$$

Proof. Firstly, by Lemma 4.3, we have

$$\begin{aligned} \dot{x}_2(t) - \dot{x}_1(t) &= (c_2^0 - c_1^0) + (\dot{x}_2(t) - c_2(t)) - (\dot{x}_1(t) - c_1(t)) \\ &\quad + (c_2(t) - c_2(0)) - (c_1(t) - c_1(0)) + (c_2(0) - c_2^0) - (c_1(0) - c_1^0) \\ &\geq \frac{c_2^0 - c_1^0}{4}, \end{aligned}$$

therefore, integrating in $t > 0$ gives us that for $t > 0$

$$x_2(t) - x_1(t) \geq \frac{L}{2} + \frac{c_2^0 - c_1^0}{4} t \geq \frac{L}{2} + 8\theta_2 t.$$

Thus,

$$\begin{aligned} \int |\mathcal{R}_1(t, x) \mathcal{R}_2(t, x)| \, dx &\leq C \int e^{-\frac{\sqrt{4\omega_1(t) - c_1^2(t)}}{2} |x - x_1(t)|} e^{-\frac{\sqrt{4\omega_2(t) - c_2^2(t)}}{2} |x - x_2(t)|} \, dx \\ &\leq C \int e^{-\frac{\sqrt{4\omega_1^0 - (c_1^0)^2}}{4} |x - x_1(t)|} e^{-\frac{\sqrt{4\omega_2^0 - (c_2^0)^2}}{4} |x - x_2(t)|} \, dx \\ &\leq C e^{-\frac{\sqrt{4\omega_1^0 - (c_1^0)^2}}{8} |x_2(t) - x_1(t)|} \\ &\leq C e^{-8\theta_2(\frac{L}{2} + 8\theta_2 t)}. \end{aligned}$$

Secondly, as for (B.2), we only estimate the former term since the later term can be proved in the same way. By Lemma 4.3, we have for sufficiently small α_0 and sufficiently large L_0

$$\frac{d}{dt} (\bar{x}^0 + \sigma t - \sqrt{t + a} - x_1(t)) = \sigma - \frac{1}{2\sqrt{t + a}} - \dot{x}_1(t)$$

$$\begin{aligned}
&\geq (\sigma - c_1^0) - \frac{4}{L} - (\dot{x}_1(t) - c_1(t)) + (c_1^0 - c_1(t)) \\
&\geq \frac{\sigma - c_1^0}{4},
\end{aligned} \tag{B.3}$$

By integrating with respect to t , we obtain

$$\begin{aligned}
\bar{x}^0 + \sigma t - \sqrt{t+a} - x_1(t) &\geq \bar{x}^0 - \frac{L}{8} - x_1(0) + \frac{\sigma - c_1^0}{4} t \\
&\geq \frac{L}{4} + 4\theta_2 t.
\end{aligned} \tag{B.4}$$

This implies that

$$\begin{aligned}
\int |\mathcal{R}_1(t, x) \mathfrak{h}(t, x)| \, dx &\leq C \int_{x > \bar{x}^0 + \sigma t - \sqrt{t+a}} e^{-\frac{\sqrt{4\omega_1(t) - c_1^2(t)}}{2} |x - x_1(t)|} \, dx \\
&\leq C \int_{x > \bar{x}^0 + \sigma t - \sqrt{t+a}} e^{-\frac{\sqrt{4\omega_1^0 - (c_1^0)^2}}{4} |x - x_1(t)|} \, dx \\
&\leq C e^{-16\theta_2(\frac{L}{4} + 4\theta_2 t)}.
\end{aligned}$$

This ends the proof. \square

Proof of Lemma 6.1. We now expand $\mathfrak{E}(u(t))$ one by one.

The term: $\int |\partial_x u(t)|^2 \, dx$. By (B.1) and integration by parts, we have

$$\begin{aligned}
\int |\partial_x u(t)|^2 \, dx &= \sum_{k=1}^2 \int |\partial_x R_k(t)|^2 \, dx - \sum_{k=1}^2 2\Re \int \partial_x^2 R_k(t) \bar{\varepsilon}(t) \, dx + \int |\partial_x \varepsilon(t)|^2 \, dx \\
&\quad + O\left(e^{-8\theta_2(\frac{L}{2} + 8\theta_2 t)}\right).
\end{aligned}$$

The term: $\int |u(t)|^6 \, dx$. By (B.1) and Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
\int |u(t)|^6 \, dx &= \sum_{k=1}^2 \int |R_k(t)|^6 \, dx + \sum_{k=1}^2 \int 6 |R_k(t)|^4 \Re(R_k \bar{\varepsilon})(t) \, dx \\
&\quad + \sum_{k=1}^2 \int 3 |R_k(t)|^4 |\varepsilon(t)|^2 + 12 |R_k(t)|^2 [\Re(R_k \bar{\varepsilon})(t)]^2 \, dx \\
&\quad + \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \beta \left(\|\varepsilon(t)\|_{H^1(\mathbb{R})} \right) + O\left(e^{-8\theta_2(\frac{L}{2} + 8\theta_2 t)}\right).
\end{aligned}$$

The term: $\frac{\omega_1(0)}{2} \int |u(t)|^2 \mathfrak{g}(t) \, dx$ and $\frac{\omega_2(0)}{2} \int |u(t)|^2 \mathfrak{h}(t) \, dx$. By (B.2) and the Cauchy-Schwarz inequality, we have

$$\frac{\omega_1(0)}{2} \int |u(t)|^2 \mathfrak{g}(t) \, dx$$

$$\begin{aligned}
&= \frac{\omega_1(0)}{2} \int \left| \sum_{k=1}^2 R_k(t) + \varepsilon(t) \right|^2 \mathfrak{g}(t) \, dx \\
&= \frac{\omega_1(0)}{2} \int |R_1(t)|^2 - |R_1(t)|^2 \mathfrak{h}(t) + |R_2(t)|^2 \mathfrak{g}(t) + 2\Re(R_1 \overline{R_2})(t) \mathfrak{g}(t) \, dx \\
&\quad + \frac{\omega_1(0)}{2} \int \Re(R_1 \overline{\varepsilon}) - \Re(R_1 \overline{\varepsilon})(t) \mathfrak{h}(t) + \Re(R_2 \overline{\varepsilon})(t) \mathfrak{g}(t) \, dx \\
&\quad + \frac{\omega_1(0)}{2} \int |\varepsilon(t)|^2 \mathfrak{g}(t) \, dx \\
&= \frac{\omega_1(0)}{2} \int |R_1(t)|^2 \, dx + \frac{\omega_1(0)}{2} \int \Re(R_1 \overline{\varepsilon}) \, dx + \frac{\omega_1(0)}{2} \int |\varepsilon(t)|^2 \mathfrak{g}(t) \, dx \\
&\quad + O\left(\int |R_1(t)|^2 \mathfrak{h}(t) + |R_2(t)|^2 \mathfrak{g}(t) + |R_1(t)| |R_2(t)| \mathfrak{g}(t) \, dx \right) \\
&\quad + O\left(\int |R_1(t)| |\varepsilon(t)| \mathfrak{h}(t) + |R_2(t)| |\varepsilon(t)| \mathfrak{g}(t) \, dx \right) \\
&= \frac{\omega_1(0)}{2} \int |R_1(t)|^2 \, dx + \frac{\omega_1(0)}{2} \int \Re(R_1 \overline{\varepsilon}) \, dx + \frac{\omega_1(t)}{2} \int |\varepsilon(t)|^2 \mathfrak{g}(t) \, dx \\
&\quad + \frac{\omega_1(0) - \omega_1(t)}{2} \int |\varepsilon(t)|^2 \mathfrak{g}(t) \, dx \\
&\quad + O\left(\int |R_1(t)|^2 \mathfrak{h}(t) + |R_2(t)|^2 \mathfrak{g}(t) + |R_1(t)| |R_2(t)| \mathfrak{g}(t) \, dx \right) \\
&\quad + O\left(\int |R_1(t)| |\varepsilon(t)| \mathfrak{h}(t) + |R_2(t)| |\varepsilon(t)| \mathfrak{g}(t) \, dx \right) \\
&= \frac{\omega_1(0)}{2} \int |R_1(t)|^2 \, dx + \frac{\omega_1(0)}{2} \int \Re(R_1 \overline{\varepsilon}) \, dx + \frac{\omega_1(t)}{2} \int |\varepsilon(t)|^2 \mathfrak{g}(t) \, dx \\
&\quad + \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \beta \left(\|\varepsilon(t)\|_{H^1(\mathbb{R})} \right) + O\left(|\omega_1(t) - \omega_1(0)|^2 \right) + O\left(e^{-8\theta_2(\frac{L}{2} + 8\theta_2 t)} \right),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\omega_2(0)}{2} \int |u(t)|^2 \mathfrak{h}(t) \, dx \\
&= \frac{\omega_2(0)}{2} \int |R_2(t)|^2 \, dx + \frac{\omega_2(0)}{2} \int \Re(R_2 \overline{\varepsilon}) \, dx + \frac{\omega_2(t)}{2} \int |\varepsilon(t)|^2 \mathfrak{g}(t) \, dx \\
&\quad + \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \beta \left(\|\varepsilon(t)\|_{H^1(\mathbb{R})} \right) + O\left(|\omega_2(t) - \omega_2(0)|^2 \right) + O\left(e^{-8\theta_2(\frac{L}{2} + 8\theta_2 t)} \right).
\end{aligned}$$

The term $-\frac{c_1(0)}{2} \Im \int (\overline{u} \partial_x u)(t) \mathfrak{g}(t) \, dx$ **and** $-\frac{c_2(0)}{2} \Im \int (\overline{u} \partial_x u)(t) \mathfrak{h}(t) \, dx$. Similarly, we have

$$\begin{aligned}
&-\frac{c_1(0)}{2} \Im \int (\overline{u} \partial_x u)(t) \mathfrak{g}(t) \, dx \\
&= -\frac{c_1(0)}{2} \Im \int \overline{R_1} \partial_x R_1(t) \, dx - c_1(0) \Im \int \partial_x R_1(t) \overline{\varepsilon} \, dx - \frac{c_1(t)}{2} \Im \int \overline{\varepsilon} \partial_x \varepsilon(t) \mathfrak{g}(t) \, dx \\
&\quad + \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \beta \left(\|\varepsilon(t)\|_{H^1(\mathbb{R})} \right) + O\left(|c_1(t) - c_1(0)|^2 \right) + O\left(e^{-8\theta_2(\frac{L}{2} + 8\theta_2 t)} \right),
\end{aligned}$$

and

$$\begin{aligned}
& -\frac{c_2(0)}{2} \Im \int (\bar{u} \partial_x u)(t)(t) \mathfrak{h}(t) \, dx \\
& = -\frac{c_2(0)}{2} \Im \int \bar{R}_2 \partial_x R_2(t) \, dx - c_2(0) \Im \int \partial_x R_2(t) \bar{\varepsilon} \, dx - \frac{c_2(t)}{2} \Im \int \bar{\varepsilon} \partial_x \varepsilon(t) \mathfrak{h}(t) \, dx \\
& \quad + \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \beta \left(\|\varepsilon(t)\|_{H^1(\mathbb{R})} \right) + O \left(|c_2(t) - c_2(0)|^2 \right) + O \left(e^{-8\theta_2(\frac{L}{2} + 8\theta_2 t)} \right).
\end{aligned}$$

The term $\frac{c_1(0)}{8} \int |u(t)|^4 \mathfrak{g}(t) \, dx$ **and** $\frac{c_2(0)}{8} \int |u(t)|^4 \mathfrak{h}(t) \, dx$.

$$\begin{aligned}
& \frac{c_1(0)}{8} \int |u(t)|^4 \mathfrak{g}(t) \, dx \\
& = \frac{c_1(0)}{8} \int |R_1(t)|^4 \, dx + \frac{c_1(0)}{4} \int |R_1(t)|^2 \Re(R_1 \bar{\varepsilon})(t) \, dx \\
& \quad + \frac{c_1(t)}{8} \int 2|R_1(t)|^2 |\varepsilon(t)|^2 + [\Re(R_1 \bar{\varepsilon})(t)]^2 \, dx \\
& \quad + \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \beta \left(\|\varepsilon(t)\|_{H^1(\mathbb{R})} \right) + O \left(|c_1(t) - c_1(0)|^2 \right) + O \left(e^{-8\theta_2(\frac{L}{2} + 8\theta_2 t)} \right),
\end{aligned}$$

and

$$\begin{aligned}
& \frac{c_2(0)}{8} \int |u(t)|^4 \mathfrak{h}(t) \, dx \\
& = \frac{c_2(0)}{8} \int |R_2(t)|^4 \, dx + \frac{c_2(0)}{4} \int |R_2(t)|^2 \Re(R_2 \bar{\varepsilon})(t) \, dx \\
& \quad + \frac{c_2(t)}{8} \int 2|R_2(t)|^2 |\varepsilon(t)|^2 + [\Re(R_2 \bar{\varepsilon})(t)]^2 \, dx \\
& \quad + \|\varepsilon(t)\|_{H^1(\mathbb{R})}^2 \beta \left(\|\varepsilon(t)\|_{H^1(\mathbb{R})} \right) + O \left(|c_2(t) - c_2(0)|^2 \right) + O \left(e^{-8\theta_2(\frac{L}{2} + 8\theta_2 t)} \right).
\end{aligned}$$

Summing up the above terms, we can conclude the proof by (3.9) for $k = 1, 2$ and the orthogonal conditions (4.11). \square

C. THE COERCIVITY OF THE LOCALIZED QUADRATIC TERM

Let L be large enough, x_1 and $x_2 \in \mathbb{R}$ with $x_2 - x_1 > \frac{L}{2}$. Now, define

$$g(x) = \begin{cases} 1, & x \leq x_1 + \frac{L}{8}, \\ 0 < \cdot < 1, & x_1 + \frac{L}{8} < x < x_2 - \frac{L}{8}, \\ 0, & x \geq x_2 - \frac{L}{8}, \end{cases} \quad \text{and } h(x) = 1 - g(x). \quad (\text{C.1})$$

In order to prove Lemma 6.2, it suffices to show the following result.

Lemma C.1. *Let $L > 1$ be large enough, g, h be given by (C.1). Then there exists $C_1 > 0$ such that*

$$\mathcal{H}_2(\varepsilon, \varepsilon) \geq C_1 \|\varepsilon\|_{H^1(\mathbb{R})},$$

where

$$\begin{aligned} \mathcal{H}_2(\varepsilon, \varepsilon) = & \frac{1}{2} \int |\varepsilon_x|^2 - \frac{1}{32} \left(3 \int |R_1|^4 |\varepsilon|^2 + 12 \int |R_1|^2 [\Re(\overline{R_1} \varepsilon)]^2 \right) \\ & - \frac{1}{32} \left(3 \int |R_2|^4 |\varepsilon|^2 + 12 \int |R_2|^2 [\Re(\overline{R_2} \varepsilon)]^2 \right) \\ & + \frac{\omega_1}{2} \int |\varepsilon|^2 g - \frac{c_1}{2} \Im \int \overline{\varepsilon} \varepsilon_x g + \frac{c_1}{8} \left(2 \int |R_1|^2 |\varepsilon|^2 + 4 \int [\Re(\overline{R_1} \varepsilon)]^2 \right) \\ & + \frac{\omega_2}{2} \int |\varepsilon|^2 h - \frac{c_2}{2} \Im \int \overline{\varepsilon} \varepsilon_x h + \frac{c_2}{8} \left(2 \int |R_2|^2 |\varepsilon|^2 + 4 \int [\Re(\overline{R_2} \varepsilon)]^2 \right), \end{aligned}$$

with $R_k(x) = \varphi_{\omega_k, c_k}(x - x_k) e^{i\gamma_k}$ ($k = 1, 2$).

First, we give a localized version of the ‘single solitary’ coercive result. For the convenience of notation, we denote

$$R(x) = \varphi_{\omega, c}(x - y_0) e^{i\gamma},$$

with $4\omega > c^2$, $y_0, \theta \in \mathbb{R}$. Let $\Phi : \mathbb{R} \mapsto \mathbb{R}$ be an even C^2 function with

$$\Phi(x) = \begin{cases} 1, & |x| \leq 1, \\ e^{-|x|} \leq \cdot \leq 3e^{-|x|}, & 1 < |x| < 2, \\ e^{-|x|}, & |x| \geq 2, \end{cases}$$

and $\Phi'(x) \leq 0$ for $x > 0$.

Lemma C.2. *Let $B > 1$ be large enough. If $\varepsilon \in H^1(\mathbb{R})$ satisfies the following orthogonality condition,*

$$\begin{aligned} \Re \int R(x) \overline{\varepsilon(x)} dx &= 0, \quad \Re \int \left(i\partial_x R + \frac{1}{2} |R|^2 R \right)(x) \overline{\varepsilon(x)} dx = 0, \\ \Re \int \partial_x R(x) \overline{\varepsilon(x)} dx &= 0, \quad \Re \int iR(x) \overline{\varepsilon(x)} dx = 0. \end{aligned}$$

Then, we have

$$\mathcal{H}_{B, y_0}(\varepsilon, \varepsilon) \geq \frac{C_0}{4} \int (|\varepsilon_x|^2 + |\varepsilon|^2) \Phi_{B, y_0} dx,$$

where

$$\begin{aligned} \mathcal{H}_{B, y_0}(\varepsilon, \varepsilon) = & \frac{1}{2} \int |\varepsilon_x|^2 \Phi_{B, y_0} dx + \frac{\omega}{2} \int |\varepsilon|^2 \Phi_{B, y_0} dx - \frac{c}{2} \Im \int \overline{\varepsilon} \varepsilon_x \Phi_{B, y_0} dx \\ & + \frac{c}{8} \left(2 \int |R|^2 |\varepsilon|^2 \Phi_{B, y_0} dx + 4 \int [\Re(\overline{R} \varepsilon)]^2 \Phi_{B, y_0} dx \right) \\ & - \frac{1}{32} \left(3 \int |R|^4 |\varepsilon|^2 \Phi_{B, y_0} dx + 12 \int |R|^2 [\Re(\overline{R} \varepsilon)]^2 \Phi_{B, y_0} dx \right). \end{aligned}$$

Proof. By setting $\zeta(x) = \sqrt{\Phi_{B,y_0}(x)}\varepsilon(x)$, we have

$$\begin{aligned} |\varepsilon_x|^2 \Phi_{B,y_0} &= |\zeta_x|^2 - \frac{\Phi'_{B,y_0}}{\Phi_{B,y_0}} \Re(\bar{\zeta} \zeta_x) + \frac{1}{4} \left(\frac{\Phi'_{B,y_0}}{\Phi_{B,y_0}} \right)^2 |\zeta|^2. \\ \Im(\bar{\varepsilon} \varepsilon_x) \Phi_{B,y_0} &= \Im(\bar{\zeta} \zeta_x), \quad \text{and} \quad |\varepsilon|^2 \Phi_{B,y_0} = |\zeta|^2. \end{aligned}$$

Now, we rewrite the quadratic form \mathcal{H}_{B,y_0} as a quadratic form with respect to ζ , which means

$$\mathcal{H}_{B,y_0}(\varepsilon, \varepsilon) = \mathcal{H}_{\omega,c}(\zeta, \zeta) - \frac{1}{2} \Re \int \frac{\Phi'_{B,y_0}}{\Phi_{B,y_0}} \bar{\zeta} \zeta_x \, dx - \frac{1}{4} \int \left(\frac{\Phi'_{B,y_0}}{\Phi_{B,y_0}} \right)^2 |\zeta|^2 \, dx.$$

On the one hand, as a consequence of Lemma 3.2, we obtain

$$\begin{aligned} \mathcal{H}_{\omega,c}(\zeta, \zeta) &\geq \frac{C_0}{2} \|\zeta\|_{H^1(\mathbb{R})}^2 - \frac{2}{C_0} \left[\left(\Re \int R \bar{\zeta} \, dx \right)^2 + \left(\Re \int \left(i \partial_x R + \frac{1}{2} |R|^2 R \right) \bar{\zeta} \, dx \right)^2 \right] \\ &\quad - \frac{2}{C_0} \left[\left(\Re \int \partial_x R \bar{\zeta} \, dx \right)^2 + \left(\Re \int i R \bar{\zeta} \, dx \right)^2 \right]. \end{aligned}$$

On the other hand, a straight calculation implies that

$$\begin{aligned} \left| \Re \int R \bar{\zeta} \, dx \right| &= \left| \Re \int R \bar{\varepsilon} \left(1 - \sqrt{\Phi_{B,y_0}} \right) \, dx \right| \\ &= \left| \Re \int_{|x-y_0|>B} R \bar{\varepsilon} \left(1 - \sqrt{\Phi_{B,y_0}} \right) \, dx \right| \\ &\leq \|\varepsilon\|_2 \left(\int_{|x-y_0|>B} |R|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq \frac{C}{e^{\sqrt{4\omega-c^2} \frac{B}{2}}} \|\varepsilon\|_2, \end{aligned}$$

moreover, applying the similar argument to $\Re \int \left(i \partial_x R + \frac{1}{2} |R|^2 R \right) \bar{\zeta}$, $\Re \int \partial_x R \bar{\zeta}$ and $\Re \int i R \bar{\zeta}$ gives us that

$$\mathcal{H}_{\omega,c}(\zeta, \zeta) \geq \frac{C_0}{2} \|\zeta\|_{H^1(\mathbb{R})}^2 - \frac{C}{e^{\sqrt{4\omega-c^2} B}} \|\zeta\|_2^2.$$

Now it follows from $\left| \Phi'_{B,y_0} \right| \leq \frac{C}{B} \Phi_{B,y_0}$ that, for $B > 1$ large enough,

$$\mathcal{H}_{B,y_0}(\varepsilon, \varepsilon) \geq \frac{C_0}{2} \|\zeta\|_{H^1(\mathbb{R})}^2 - \frac{C}{e^{\sqrt{4\omega-c^2} B}} \|\zeta\|_2^2 - \frac{C}{B^2} \|\zeta\|_{H^1(\mathbb{R})}^2 \geq \frac{3C_0}{8} \|\zeta\|_{H^1(\mathbb{R})}^2.$$

Since

$$\begin{aligned} |\zeta_x|^2 &= |\varepsilon_x|^2 \Phi_{B,y_0} + \frac{\Phi'_{B,y_0}}{\Phi_{B,y_0}} \Re(\bar{\varepsilon} \varepsilon_x) \Phi_{B,y_0} + \frac{1}{4} \left(\frac{\Phi'_{B,y_0}}{\Phi_{B,y_0}} \right)^2 |\varepsilon|^2 \Phi_{B,y_0} \\ &\geq \left(1 - \frac{C}{B^2} \right) |\varepsilon_x|^2 \Phi_{B,y_0} - \frac{C}{B^2} |\varepsilon|^2 \Phi_{B,y_0} \end{aligned}$$

we obtain, for B large enough,

$$\mathcal{H}_{B,y_0}(\varepsilon, \varepsilon) \geq \frac{C_0}{4} \int \left(|\varepsilon_x|^2 + |\varepsilon|^2 \right) \Phi_{B,y_0} dx.$$

This ends the proof. \square

Proof of Lemma C.1. Since $L > 1$ is sufficiently large enough, we can take $B \in (1, \frac{L}{4})$ such that Claim C.2 holds.

$$\begin{aligned} \mathcal{H}_2(\varepsilon, \varepsilon) &= \mathcal{H}_{B,x_1}(\varepsilon, \varepsilon) + \mathcal{H}_{B,x_2}(\varepsilon, \varepsilon) \\ &+ \frac{1}{2} \int \left[|\varepsilon_x|^2 + \omega_1 |\varepsilon| - c_1 \Im(\bar{\varepsilon} \varepsilon_x) \right] (g - \Phi_{B,x_1}) dx \\ &+ \frac{1}{2} \int \left[|\varepsilon_x|^2 + \omega_2 |\varepsilon| - c_2 \Im(\bar{\varepsilon} \varepsilon_x) \right] (h - \Phi_{B,x_2}) dx \\ &+ \frac{c_1}{8} \left(2 \int |R_1|^2 |\varepsilon|^2 (1 - \Phi_{B,x_1}) dx + 4 \int [\Re(\bar{R}_2 \varepsilon)]^2 (1 - \Phi_{B,x_1}) dx \right) \\ &+ \frac{c_2}{8} \left(2 \int |R_2|^2 |\varepsilon|^2 (1 - \Phi_{B,x_2}) dx + 4 \int [\Re(\bar{R}_1 \varepsilon)]^2 (1 - \Phi_{B,x_2}) dx \right) \\ &- \frac{1}{32} \left(3 \int |R_1|^4 |\varepsilon|^2 (1 - \Phi_{B,x_1}) dx + 12 \int |R_1|^2 [\Re(\bar{R}_1 \varepsilon)]^2 (1 - \Phi_{B,x_1}) dx \right) \\ &- \frac{1}{32} \left(3 \int |R_2|^4 |\varepsilon|^2 (1 - \Phi_{B,x_2}) dx + 12 \int |R_2|^2 [\Re(\bar{R}_2 \varepsilon)]^2 (1 - \Phi_{B,x_2}) dx \right). \end{aligned}$$

It follows from a direct computation that

$$\begin{aligned} g(x) - \Phi_{B,x_1}(x) &\begin{cases} = 0, & |x - x_1| < \frac{L}{8}, \\ \geq -e^{-\frac{L}{8B}}, & \text{else}, \end{cases} \\ h(x) - \Phi_{B,x_2}(x) &\begin{cases} = 0, & |x - x_2| < \frac{L}{8}, \\ \geq -e^{-\frac{L}{8B}}, & \text{else}, \end{cases} \\ 1 - \Phi_{B,x_1}(x) &\begin{cases} = 0, & |x - x_1| < \frac{L}{8}, \\ \geq -e^{-\frac{L}{8B}}, & \text{else}, \end{cases} \end{aligned}$$

and

$$1 - \Phi_{B,x_2}(x) \begin{cases} = 0, & |x - x_2| < \frac{L}{8}, \\ \geq -e^{-\frac{L}{8B}}, & \text{else}. \end{cases}$$

Moreover, since, for $k = 1, 2$, $c_k^2 < 4\omega_k$, there exists $\delta_k > 0$ such that

$$|\varepsilon_x|^2 + \omega_k |\varepsilon|^2 - c_k \Im(\bar{\varepsilon} \varepsilon_x) \geq \delta_k \left(|\varepsilon_x|^2 + |\varepsilon|^2 \right).$$

Thus, taking L large enough, we obtain

$$\begin{aligned} \mathcal{H}_2(\varepsilon, \varepsilon) &\geq \frac{C_0}{4} \int (|\varepsilon_x|^2 + |\varepsilon|^2) \Phi_{B,x_1} dx + \frac{C_0}{4} \int (|\varepsilon_x|^2 + |\varepsilon|^2) \Phi_{B,x_2} dx \\ &\quad + \delta_1 \int (|\varepsilon_x|^2 + |\varepsilon|^2) (g - \Phi_{B,x_1}) dx + \delta_2 \int (|\varepsilon_x|^2 + |\varepsilon|^2) (h - \Phi_{B,x_2}) dx \\ &\quad - C e^{-\frac{L}{4B}} \int (|\varepsilon_x|^2 + |\varepsilon|^2) dx - C e^{-\sqrt{4\omega_1 - c_1^2} \frac{L}{4}} \int |\varepsilon|^2 dx - C e^{-\sqrt{4\omega_2 - c_2^2} \frac{L}{4}} \int |\varepsilon|^2 dx \\ &\geq C_1 \|\varepsilon\|_{H^1(\mathbb{R})}^2, \end{aligned}$$

where $C_1 = \frac{1}{2} \min \left\{ \frac{C_0}{4}, \delta_1, \delta_2 \right\}$. This concludes the proof. \square

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